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ON THE SOLITON-LIKE INTERACTIONS IN NONDISPERSIVE MEDIA

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Abstract Seymour and Varley [1] analyse certain media whose responses are governed by the nonlinear nondispersive wave equation, in which any two pulses traveling in opposite directions interact nonlinearly for a finite time when they collide but then part unaffected by the interaction. Clearer, when any two pulses are traveling in opposite directions meet and interact, they emerge from the interaction region unchanged by the interaction. This interaction is similar to those that occurs when two solitons collide. The main difference is that solitons are represented by waves of permanent form whose profiles are specific. The waves described by Seymour and Varley distort as they propagate, and are of arbitrary shape and amplitude. Since such media transmit waves that *do not remember the interaction process*, they are called *DRIP media*.

Key words: Wave motion, Interaction of waves, Solitons, DRIP media.

1. INTRODUCTION

In this paper we analyse the properties of a system of coupled wave equations

$$y_{tt} = A^2(y_x)y_{xx}, \tag{1}$$

$$\frac{dA}{dy_x} = A^{3/2}(\mu + \nu A), \tag{2}$$

that govern the motion of an heterogeneous string, where $y(x,t)$ is the physical displacement, $A(y_x)$ a positive function representing the local speed of propagation, and μ, ν the material constants. If $y_x = y_x(x)$ and $A(y_x(x)) = A(x)$, the above system of equations can be written under the form

$$y_{tt} = A^2(x)y_{xx}, \tag{3}$$

$$A_x = A^{3/2}(\mu + \nu A)y_{xx}, \tag{4}$$

We show that:
a. The waves described by (3) and (4) are dispersive and dissipative.
b. The single bounded solution of (2) is given by

$$A(y_x) = \frac{\lambda[e_3 + (e_2 - e_3)\mathrm{sn}^2(\sqrt{e_1 - e_3}\,y_x + \delta')]}{1 + \rho[e_3 + (e_2 - e_3)\mathrm{sn}^2(\sqrt{e_1 - e_3}\,y_x + \delta')]}, \tag{5}$$

with λ,ρ constants depending on μ,ν, and e_i, $i=1,2,3$ the solutions of the equation $4y^3-g_2y-g_3=0$, with constants g_2,g_3 depending on μ,ν,

The solution (5) shows a cnoidal dependence of A on y_x. It is characterized by the dependence of the amplitude on the argument of sn.

For a certain value of μ,ν, for that $m=1$, ($m=\frac{e_2-e_3}{e_1-e_3}$) the solution (6) becomes

$$A(y_x)=\frac{\lambda[e_1-(e_1-e_3)\mathrm{sech}^2(\sqrt{e_1-e_3}\,y_x+\delta')]}{1+\rho[e_1-(e_1-e_3)\mathrm{sech}^2(\sqrt{e_1-e_3}\,y_x+\delta')]}. \tag{6}$$

This solution (6) shows a solitonic dependence of A on y_x. The solution (5) is also characterized by the dependence of the amplitude on the argument of sech.

The *interaction* (collision) of two solutions such (5) or (6) has the solitonic properties: may *propagate* without change of form, being regarded as a local confinement of the energy of the wave field. At the collision each may come away with the same character as it had before the collision [2-4].

c. The solutions of (3) depend on A given by (5) or (6). Any two solutions of (3) traveling in opposite directions interact nonlinearly and the collision is certainly influenced by the properties of A. The conclusion is the solutions of (3) and (4) are expressed in terms of cnoidal or soliton functions and their interaction have the cnoidal or solitonic properties.

The case of soliton-like interaction described by Seymour and Varley [1] in the section 4 and 5 is explained due to a solitonic behaviour of A. For a cnoidal behavior of A we do not know what happens. We must analyse the collision phenomenon in this case [2-4].

2. THE LINEAR VIBRATING STRING

Consider 1D string motion equation [5, 6]

$$u_{tt}-c^2u_{xx}=0, \tag{7}$$

with c a real positive number.

Let x range from $-\infty$ to ∞. For the transverse vibrations of a string $c^2=T/\lambda$ where T is the constant tension and λ the mass per unit length at the position x. For the compressional vibrations of an isotropic elastic solid in which the density and elastic constants are functions of x only (laminated medium) $c^2=(\lambda+2\mu)/\rho$. For the transverse vibrations of such laminated solid $c^2=\mu/\rho$. The characteristics are given by $\frac{dx}{dt}=\pm c$, that are straight lines inclined to the axis at $c=\tan\varphi$. The D'Alembert solution of (7) is

$$u(x,t)=f(x-ct)+g(x+ct), \tag{8}$$

where the functions $f,g:R\to R$ are determined from the initial conditions attached to (7)

$$u(x,0) = \Phi(x), \qquad \frac{\partial u}{\partial t}(x,0) = \Psi(x)\,. \tag{9}$$

We have

$$f(x) = \frac{1}{2}\Phi(x) + \frac{1}{2c}\int_0^x \Psi(\alpha)\mathrm{d}\alpha + \frac{a}{2}, \tag{10}$$

$$g(x) = \frac{1}{2}\Phi(x) - \frac{1}{2c}\int_0^x \Psi(\alpha)\mathrm{d}(\alpha) + \frac{a}{2}, \tag{11}$$

with a a real constant.

The solution (8) describes two waves $f(x-ct)$ and respectively $g(x+ct)$.

Geometrically, the function $u(x,t)$ can be represented as a surface in the space (u,x,t). A section through this surface in the plane $t = t_0$, is $u = u(x,t_0)$ and represents the profile of the vibrating string (a wave) at the time $t = t_0$. A section through the surface in the plane $x = x_0$, is $u = u(x_0,t)$ and represents the motion phenomenon of the point x_0.

The modified profiles of $f(x-ct)$ can be determined in the following way. Consider one observer with a system of coordinates (x',t') so that, at the time $t_0 = 0$, the observer occupies the position $x = 0$, and at the time t, the position ct, since it has a rectilinear motion with the velocity c. In the new coordinate system (x',t'), attached to the observer $(t' = t, x' = x - ct)$, the function $f(x-ct)$ is given , at any time t' , by $f(x')$. The observer sees, at any moment of time the unchanged profile $f(x)$ at the initial time $t_0 = t'_0 = 0$. This is way the function $f(x-ct)$ represents a right travelling wave or a forward-going wave with the velocity *c*. For a similar reason, $g(x+ct)$ represents a left travelling wave or a backward-going wave with the velocity c.

As a consequence, both waves are not interacting between them and do not change their shape during the propagation. These waves can be superposed by a simple sum, because of the linearity of (7). These waves can be called *solitary waves*, for the reason they are not changing their shapea during propagation process, and do not interact one with the other (Fig. 1).

If (7) is not linear (the case considered by Seymour and Varley [1]

$$y_{tt} = A^2(y_x)y_{xx}, \tag{12}$$

the superposition principle is not valid.

If y_1, y_2 are two solutions of (12), the sum between them is not a solution of (12).

The function $A(y_x(x))$ is a positive function that represents the local speed of the propagation.

In the last years some nonlinear equations are intensive studied [2]

Korteveg de Vries (KdV) equation $u_t - 6uu_x + u_{xxx} = 0$;

Boussinesq equation $u_{xxxx} + 24(u_x^{\,2} + uu_{xx}) + 12u_{xx} - 12u_{tt} = 0$;

Burgers equation $u_t + uu_x - u_{xx} = 0$;

Sine Gordon equation $u_{xx} - u_{tt} = \sin u$;

Nonlinear Schrodinger equation $iu_t + u_{xx} \pm u|u|^2 = 0$.

These equations admit solutions similarly with the waves described before, but having new properties that made the waves to have a particle behaviour: the waves are localized, bounded, and tend in time to a constant. They can interact one to another without changing their identity (amplitude, velocity, shape). These special solutions are called *solitons*.

The notion of soliton has appeared for the first time by Zabusky and Kruskal [7] in1965.

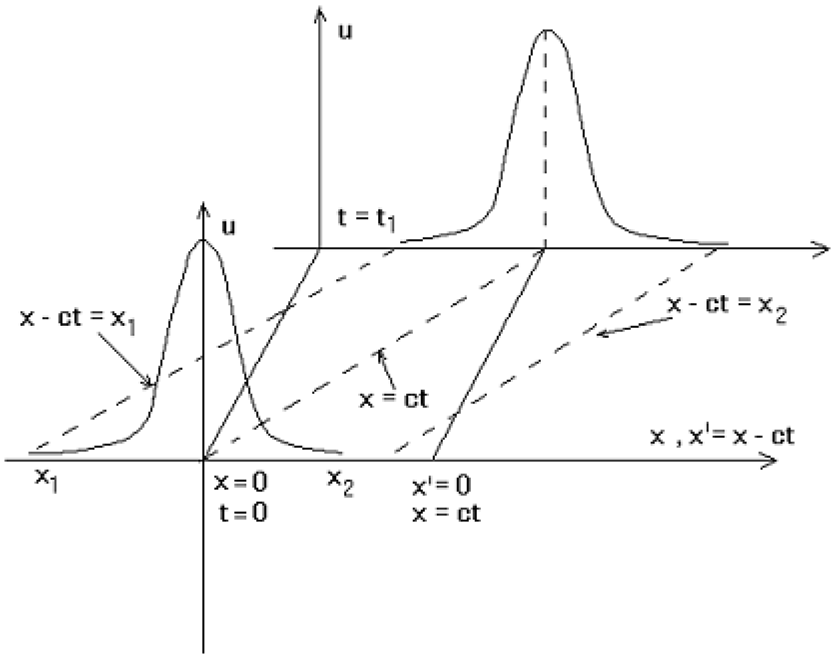

Fig. 1. Geometrical significance of the solitary wave.

In the above paper, the authors have solved numerically the KdV equation, with conditions $u \to 0$ at $|x| \to \infty$, and initial conditions given by two solitary waves, separated in space, that propagate in the same direction. In this way the authors obtain the characteristics of two soliton collision.

If a nonlinear equation admits a solution expressed by a single solitary wave, it can be named soliton. But, if the equation admits as solutions more solitary waves, we do not know if these solutions are solitons. They can collide and after collision their shape can be modified by appearing some extra oscillations.

3. DISPERSION AND DISSIPATION OF THE HARMONIC WAVES

Let consider a harmonic wave

$$v(x,t) = \tilde{A} \exp \mathrm{i}(kx - \omega t). \tag{13}$$

representing the solution of (7).

The real number k is wave-number, the complex number ω is the frequency, $\lambda = \frac{2\pi}{k}$ is wavelength, and the real number $\tilde{A}$, the amplitude.

The phase velocity is the speed of the phase $\phi = \omega t - kx$, it is the velocity of propagation of a surface with constant phase

$$c_p = \frac{\omega}{k}, \tag{14}$$

the group velocity is the speed of a volume

$$c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k}. \tag{15}$$

Introducing (13) into (7) we obtain the dispersion relation

$$F(\omega, k) = 0. \tag{16}$$

If ω is real, we say that have the *dispersion of waves*, if the phase velocity depends on the wave-number. In the case of the linearised KdV equation

$$u_t - 6u_x + u_{xxx} = 0, \tag{17}$$

the dispersion relation is

$$\omega + k + k^3 = 0. \tag{18}$$

When $\mathrm{Re}\frac{\omega}{k} = \text{const.}$ the waves are nondispersive, and for $\mathrm{Im}\frac{\omega}{k} = 0$ the waves are nondissipative.

The phase velocity $c_p = -1 - k^2$ depends on the wave-number and then we have the dispersion phenomenon. The group velocity $c_g = -1 - 3k^2$ differs by the phase velocity for $k \neq 0$, and in consequence the components of waves scatter and disperse in the propagation process.

Ife ω is complex $\omega = \mathrm{Re}\,\omega + i\,\mathrm{Im}\,\omega,\ \mathrm{Im}\,\omega \lhd 0$, we say we have *dissipation of waves.*

The solution in this case is

$$v(x,t) = \tilde{A}\exp(t\,\mathrm{Im}\,\omega)\exp \mathrm{i}(kx - \mathrm{Re}\,\omega t), \tag{19}$$

and the amplitude is exponential decreasing at $t \to \infty$.

For the linearised Burgers equation

$$u_t + u_x + u_{xx} = 0, \tag{20}$$

the dispersion relation is

$$\omega = k - \mathrm{i}k^2. \tag{21}$$

The phase velocity of the harmonic waves

$$v(x,t) = \tilde{A}\exp(-tk^2)\exp \mathrm{i}k(x - t),$$

is $c_p = 1$, and the group velocity, $c_g = 1 - 2ik$.

The dissipation appears because $\mathrm{Im}\,\omega = -k^2$ is negative for any real k.

In conclusion, the KdV equation is dispersive due to the term u_{xxx}, and Burgers equation is dissipative due to u_{xx}.

3. THE NONLINEAR VIBRATING STRING

Consider the equation (12)

$$y_{tt} = A^2(y_x)y_{xx}\,, \tag{22}$$

that can governs the motion of an heterogeneous string, where y is the physical displacement, and $A(y_x)$ is a positive function representing the local speed of propagation and verifies

$$\frac{\mathrm{d}A}{\mathrm{d}y_x} = A^{3/2}(\mu + \nu A)\,. \tag{23}$$

Since

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{\mathrm{d}A}{\mathrm{d}y_x}\frac{\mathrm{d}y_x}{\mathrm{d}x} = A^{3/2}(\mu + \nu A)y_{xx}\,, \tag{24}$$

we obtain a coupled partial nonlinear differential equations for $y(x,t)$ and $A(x)$

$$y_{tt} = A^2(x)y_{xx}\,, \tag{25}$$

$$A_x = A^{3/2}(\mu + \nu A)y_{xx}\,. \tag{26}$$

The characteristics are given by $\frac{\mathrm{d}x}{\mathrm{d}t} = \pm A$, which equations define two congruences of curves in the (x,t) plane.

1) First step is to straighten the characteristics of (25) for geometrical representation in a space-time plane. To do this we define the transformation [5, 6]

$$x \to u(x) = \int_0^x \frac{\mathrm{d}z}{A(z)}\quad . \tag{27}$$

2) From (3.4) we obtain

3) $y_{tt} - y_{uu} + \frac{c_u}{c} y_u = 0$, $c(u(x)) = A(x)$. (28)

4) The function $c(u)$ is the transformed local speed.

5) We have taking into account that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y_u \frac{1}{c},$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = y_{uu}\frac{1}{c^2} - y_u \frac{c_y}{c^2}\frac{1}{c}.$$

Strictly speaking we should not use the same symbols y in (28), because the function $y(x,t)$ of (25)-(26) is not the same function as the $y(u,t)$ of (28), but this is not likely to cause confusion if we remember that y may be regarded as a physical quantity in terms of (x,t) or (u,t).

The linearized form of (28) is

$$y_{tt} - y_{uu} + y_u = 0 \quad . \tag{29}$$

Introducing the harmonic wave $y(u,t) = \tilde{A}\exp \mathrm{i}(ku - \omega t)$ into (29), we obtain the dispersion relation

$$\omega^2 = \mathrm{i}k + k^2, \tag{30}$$

or

$$\frac{\omega}{k} = \sqrt{1 + \frac{\mathrm{i}}{k}} = \frac{1}{2kb} + \mathrm{i}b, \quad b = -\sqrt{\frac{-k + \sqrt{1+k^2}}{2k}}, \tag{31}$$

The phase velocity of the harmonic waves

$$y(u,t) = \tilde{A}\exp(tkb)\exp \mathrm{i}k(u - \frac{t}{2kb}),$$

is

$$c_p = \frac{1}{2kb} = \frac{-1}{\sqrt{2k(-k + \sqrt{1+k^2})}}$$

and depends on k.

In conclusion, the equation (25) is *dispersive and dissipative.*

In a space-time plane in which u and t are Cartesian coordinates, the characteristics are $\frac{\mathrm{d}x}{\mathrm{d}t} = \pm 1$, and are straight lines inclined to the axis at 45^0. Let show this more clearly.

We can change the variable and obtain for (28) another form. Let

$$y(u,t) = v(u,t)w(u), \tag{32}$$

where w is unspecified.

Then

$$y_{tt} = v_{tt}w\,,\ \ y_u = v_u w + vw'\,,\ \ y_{uu} = v_{uu}w + 2v_u w' + vw''\,.$$

The eq. (3.7) becomes, on division by w

$$v_{tt} - v_{uu} = 2kv_u + 2hv\,, \tag{33}$$

where

$$2k = \frac{2w'}{w} - \frac{c'}{c}\,,\quad 2h = \frac{w''}{w} - \frac{w'/w}{c'/c}$$

Let choose w to have $k = 0$

$$w = \sqrt{c}\,,\quad y = v\sqrt{c}\,. \tag{34}$$

Then

$$2h = -\frac{1}{2}\frac{c''}{c} + \frac{3}{4}(\frac{c'}{c})^2\ . \tag{35}$$

Equation (28) becomes

$$v_{tt} - v_{uu} = 2hv\,. \tag{36}$$

We see that the characteristics are straight lines inclined to the axis at 45^0.

4. BOUNDED SOLUTIONS OF THE EQUATION $\frac{\mathrm{d}A}{\mathrm{d}y_x} = A^{3/2}(\mu + \nu A)$

Consider the equation

$$\frac{\mathrm{d}A}{\mathrm{d}y_x} = A^{3/2}(\mu + \nu A)\,, \tag{37}$$

written under the form

$$A'^2 = A^3(\mu + \nu A)^2\,, \tag{39}$$

where $A' = \frac{\mathrm{d}A}{\mathrm{d}e}$ and $e = y_x$.

Differentiating (38), it can be written in the form

$$A'' = a_2A^2 + a_3A^3 + a_4A^4\,, \tag{40}$$

where

$$a_2 = 1.5\mu^2\,,\ \ a_3 = 4\mu\nu\,,\ \ a_4 = 2.5\nu^2\ , \tag{41}$$

and $a_i > 0\,,\ i = 2,3,4\,.$

We assume the solution of (4.3) in the form [2]

$$A = \frac{\lambda P(e)}{1+\rho P(e)}, \tag{42}$$

where $\lambda \neq 0$ and $\rho \neq 0$ are arbitrary real constants, and $P(e)$ is the Weierstrass elliptic function satisfying the differential equation [2-4, 8, 9]

$$P'^2 = 4P^3 - g_2 P - g_3, \tag{43}$$

with two invariants g_2 and g_3 which are assumed to be real and satisfy

$$g_2^3 - 27g_3^2 > 0. \tag{44}$$

Substituting (41) into (39) we obtain four equations for the four unknowns λ , ρ, g_2 and g_3

$$-2\lambda\rho^2 = a_2\lambda^2\rho^2 + a_3\lambda^3\rho + a_4\lambda^4, \tag{45}$$

$$4\lambda\rho = 2a_2\lambda^2\rho + a_3\lambda^3, \tag{46}$$

$$6\lambda + 1.5\lambda\rho^2 g_2 = a_2\lambda^2 , \tag{47}$$

$$\lambda\rho g_2 + 2\lambda\rho^2 g_3 = 0. \tag{48}$$

From (44) multiplied by 2 and (45) multiplied by ρ we have

$$\lambda^2\rho^2(2a_4\xi^2 + 3a_3\xi + 4a_2) = 0, \tag{49}$$

with $\xi = \dfrac{\lambda}{\rho}$. We are interested in real solutions of (49).

So, the condition

$$\Delta = 9a_3^2 - 32a_2a_4 > 0, \tag{50}$$

with taking account of (40), is always satisfied because $\Delta = 24\mu^2\nu^2 > 0$.

The solutions of (49) are

$$\xi_{1,2} = \frac{-3a_3 \pm 2\mu\nu\sqrt{6}}{4a_4} = \frac{\mu(-6 \pm \sqrt{6})}{5\nu} . \tag{51}$$

Therefore, there are always real valued solutions λ , ρ, g_2 and g_3 for (44)-(47) given by

$$\rho = \frac{4}{\xi(2a_2 + a_3\xi)}, \tag{52}$$

$$\lambda = \xi\rho, \tag{53}$$

$$g_2 = \frac{a_2\lambda - 6}{1.5\rho^2}, \tag{54}$$

$$g_3 = -\frac{g_2}{2\rho} . \tag{55}$$

The condition (43) becomes, by using (55) and then (54)

$$g_2^2(g_2 - \frac{27}{4\rho^2}) = g_2^2 \frac{4\mu^2 - 43}{4\rho^2} > 0 \quad . \tag{56}$$

The condition (56) is satisfied if

$$4\lambda\mu^2 - 43 > 0 \quad . \tag{57}$$

The condition (4.20) can be written under the form

$$\frac{16\mu^2}{3\mu^2 + 4\mu\nu\xi} - 43 = \frac{-113\mu^2 - 172\mu\nu\xi}{3\mu^2 + 4\mu\nu\xi} > 0 \, .$$

The numerator $-113\mu^2 - 172\mu\nu\xi$ is always positive, for both negative solutions of ξ given by (51), it is $\mu^2(59 \pm \sqrt{6}/5) > 0$. But the denominator $3\mu^2 + 4\mu\nu\xi$ is positive only for the solution $\xi = \frac{\mu(-6+\sqrt{6})}{5\nu}$, and has the value $\frac{\mu^2(-9+4\sqrt{6})}{5}$.

So, we consider only the solution

$$\xi = \frac{-3a_3 + 2\mu\nu\sqrt{6}}{4a_4} = \frac{\mu(-6+\sqrt{6})}{5\nu} \quad . \tag{58}$$

So, under the condition stated above, the exact periodic solutions can be written as

$$A(y_x) = \frac{\lambda P(y_x + \delta; g_2, g_3)}{1 + \rho P(y_x + \delta; g_2, g_3)} \quad , \tag{59}$$

where δ is an integration constant of (42), and g_2, g_3, λ and ρ are given by (52)-(55).

The exact bounded periodic solution can be obtained by replacing the Weierstrass elliptic function by the Jacobean elliptic sine function using the formula [9]

$$P(y_x + \delta; g_2, g_3) = e_3 + (e_2 - e_3)\mathrm{sn}^2(\sqrt{e_1 - e_3}\, y_x + \delta') , \tag{60}$$

where δ' is an arbitrary real constant, and e_1, e_2, e_3 are real roots of the equation

$$4y^3 - g_1 y - g_2 = 0$$

with $e_1 > e_2 > e_3$. From $cn^2 + \mathrm{sn}^2 = 1$ we can express (60) in term of the cnoidal function cn.

Thus, the exact bounded periodic solution of (39) is

$$A(y_x) = \frac{\lambda[e_3 + (e_2 - e_3)\mathrm{sn}^2(\sqrt{e_1 - e_3}\, y_x + \delta')]}{1 + \rho[e_3 + (e_2 - e_3)\mathrm{sn}^2(\sqrt{e_1 - e_3}\, y_x + \delta')]} . \tag{61}$$

5. THE SOLITON WAVE SOLUTIONS OF EQUATION $\frac{\mathrm{d}A}{\mathrm{d}y_x} = A^{3/2}(\mu + \nu A)$

The modulus m of the Jacobean elliptic function is given by

$$m = \frac{e_2 - e_3}{e_1 - e_3} \ . \tag{62}$$

The function *cn* is defined as

$$v = \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}\,, \quad \mathrm{cn}(v, m) = \mathrm{cn}\, v = \cos\varphi\,.$$

If $m = 0$ then $v = \varphi$ and cn = cos. The quantity φ is named *amplitude* of v $\varphi = am(v)$ and it is often used the notation $\mathrm{dn}v = \sqrt{1 - m\sin^2\varphi}$. For $m = 1$ we have cn v = sech v. Thus, the cnoidal solution becomes a soliton wave for $m = 1$.

The complete elliptic integral of the first kind is defined as

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}\,.$$

It is clear that $K(0) = \frac{\pi}{2}$ and $K(1) = \infty$. The period of $\cos\varphi$ is 2π, so the period of the function *cn* is $4\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}} = 4K(m)$.

The solitary wave is a periodic wave with infinite period and it is obtained when the modulus of the Jacobean elliptic function is equal to unity. So, in the soliton wave limit $e_1 = e_2$. The quantities e_1, e_2, e_3 are the roots of the equation $4y^3 - g_1 y - g_2 = 0$, $e_1 + e_2 + e_3 = 0$.

The soliton wave solutions of (39) can be written as

$$A(y_x) = \frac{\lambda[e_1 - (e_1 - e_3)\mathrm{sech}^2(\sqrt{e_1 - e_3}\, y_x + \delta')]}{1 + \rho[e_1 - (e_1 - e_3)\mathrm{sech}^2(\sqrt{e_1 - e_3}\, y_x + \delta')]} \ . \tag{63}$$

5. THE SOLUTIONS OF $y_{tt} = A^2(x) y_{xx}$

Consider the system (25) and (26) for $y(x,t)$ and $A(x)$

$$y_{tt} = A^2(x) y_{xx}\,, \tag{64}$$

$$A_x = A^{3/2}(\mu + \nu A) y_{xx}\,. \tag{65}$$

We know that $A(y_x)$ is expressed as a periodic solution of cnoidal form (61) or soliton form (63).

The amplitude $(e_1 - e_3)$ of the cnoidal solution $(e_1 - e_3)\mathrm{sn}^2(\sqrt{e_1 - e_3}\, y_x + \delta')$, or of the soliton solution $(e_1 - e_3)\mathrm{sech}^2(\sqrt{e_1 - e_3}\, y_x + \delta')$ is related to the velocity $\sqrt{e_1 - e_3}$ of the wave. Therefore, the media described by $A(y_x)$ are dispersive.

If consider that $A(y_x)$ satisfies an equation of the form (37), then $A(y_x)$ is given by (61) or (63) with λ, μ prescribed.

We show that (64, 65) predict pulses travelling in opposite directions expressed in terms of cnoidal or soliton pulses.

Consider the initial-value problem with

$$y(x,0) = f(x), \quad y_t(x,0) = g(x), \tag{66}$$

with f and g prescribed, and boundary conditions

$$A = \text{const.} \quad \text{for} \quad x \to \pm\infty \quad . \tag{67}$$

Firstly, consider the first equation (64)

$$y_{tt} = A^2(x) y_{xx} . \tag{68}$$

Let again consider the transformation (27)

$$x \to u(x) = \int_0^x \frac{\mathrm{dz}}{A(z)} \quad . \tag{69}$$

that yields to (28)

$$y_{tt} - y_{uu} + \frac{c_u}{c} y_u = 0, \quad c(u(x)) = A(x). \tag{70}$$

Equation (70) can be written in the form [5, 6]

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = L v(t), \tag{71}$$

where

$$v = \begin{pmatrix} y_u \\ y_t \end{pmatrix}, \quad L = \begin{pmatrix} 0 & \partial_u \\ \partial_u - 2\gamma & 0 \end{pmatrix}, \quad 2\gamma(u) = \frac{c_u}{c}. \tag{72}$$

We calculate $v(t)$ by a sequence of linear transformations that reduce $v(t)$ to a perturbation of the pulse [5].

For this we define the energy of the string

$$E = \frac{1}{2} \int_{-\infty}^{\infty} \frac{y_u^2 + y_t^2}{c(u)} \mathrm{d}u, \tag{73}$$

and take a point v in the phase-space (y_u, y_t)

$$v = \begin{pmatrix} q \\ p \end{pmatrix}, \quad q(u) = y_u(u,0), \quad p(u) = y_t(u,0). \tag{74}$$

The inner product is derived from the energy quadratic form (73)

$$< v_1, v_2 >= (C^{-1} v_1, C^{-1} v_2) \quad , \quad C = \begin{pmatrix} \sqrt{c(x)} & 0 \\ 0 & \sqrt{c(x)} \end{pmatrix}, \tag{75}$$

$$(v_1, v_2) = \frac{1}{2} \int_{-\infty}^{\infty} [q_1(u) q_2(u) + p_1(u) p_2(u)] du. \tag{76}$$

The operator L is screw-symmetric with respect to the inner product (75) and so there exists a one-parameter group $V(t)$ of orthogonal transformation determined by

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) = LV(t)\,, \quad V(0) = 1\,, \tag{77}$$

so that $v(t) = V(t)v$ is a solution of (71).

We base on the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(tL) = L\exp(tL)\,,$$

where

$$\exp L = E + L + \frac{L^2}{2!} + \frac{L^3}{3!} + ... = \lim_{m\to\infty}(E + \frac{L}{m})^m\,,$$

with E is the unit matrix, and $\det(\exp L) = \exp(trL)$. Since $trL = 0$, we have $\det(\exp L) = 1$.

Also we have $\exp(-t\partial x) = T(t)$ with $T(t)$ the right translation by t

$$[T(t)f](x) = f(x-t)\,,$$

and $\exp(t\partial x) = T'(t)$ with

$$T'(t) = T(-t) = f(x+t)\,,$$

the left translation by t.

The method is based on the decomposition of the phase-space (y_u, y_t) into a pair of complementary subspaces. This induces a decomposition of each initial datum into a forward propagating part and a backward-propagating part.

In the homogeneous case $(\gamma = 0)$, the equation (i) is reduced to $y_{tt} - y_{uu} = 0$ and the solution are expressed as a sum of two waves $f(x-t)$ and $f(x+t)$ that propagate independently. In the heterogeneous case the both pulses are coupled by $\gamma \neq 0$ considered as a perturbation.

So, we take

$$V(t) = CR^{-1}\tilde{V}(t)RC^{-1}\,, \;\; \tilde{V}(t) = \exp(tL) \;\; . \tag{78}$$

$$L = CR^{-1}\tilde{L}RC^{-1}\,, \;\; \tilde{L} = \begin{pmatrix} -\partial u & -\gamma \\ \gamma & \partial u \end{pmatrix}. \tag{79}$$

and

$$R = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\sqrt{2} \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{pmatrix}.$$

It results that $\tilde{V}(t) = RC^{-1}V(t)CR^{-1}$, $\tilde{L} = RC^{-1}LCR^{-1}$.

We see that $\tilde{L}$ can be written as a sum of two operators to separate the contribution of the coupling term $\gamma \neq 0$

$$\tilde{L}=\tilde{L}_0+\Gamma\,,\quad \tilde{L}_0=\begin{pmatrix}-\partial u & 0\\ 0 & \partial u\end{pmatrix},\ \ \Gamma=\begin{pmatrix}0 & -\gamma\\ \gamma & 0\end{pmatrix}. \tag{80}$$

In the homogeneous case we have $\Gamma=0$ and

$$\tilde{V}(t)=\tilde{U}(t),\quad \tilde{U}(t)=\begin{pmatrix}T(t) & 0\\ 0 & T'(t)\end{pmatrix}, \tag{81}$$

where $T(t)$ is right translation by t

$$[T(t)f](s)=f(s-t), \tag{82}$$

And $T'(t)=T(-t)$ is left translation by t.

The initial conditions (66) can be written under the form

$$y(u,0)=\phi(u)\,,\quad y_t(u,0)=\psi(u)\,. \tag{83}$$

For $\Gamma=0$ the solution of $y_{tt}-y_{uu}=0$ is written as the D'Alembert formula

$$y(u,t)=\frac{1}{2}[\phi(u+t)+\phi(u-t)]+\frac{1}{2}\int_{u-t}^{u+t}\psi(z)dz\quad . \tag{84}$$

Our aim is to obtain a similar formula for the inhomogeneous case $\Gamma\neq 0$.

For this we use the well-known perturbation formula [11]

$$\tilde{V}(t)=\tilde{U}(t)+\int_0^t \tilde{U}(t-s)\Gamma\tilde{V}(s)ds \tag{85}$$

From this we can obtain an infinite series for $\tilde{V}(t)$ by an iteration scheme

$$\tilde{V}^{(n+1)}(t)=\tilde{U}(t)+\int_0^t \tilde{U}(t-s)\Gamma\tilde{V}^{(n)}(s)ds\,,\quad \tilde{V}^{(0)}(t)=\tilde{U}(t)\,. \tag{86}$$

We take account that $\tilde{V}(t)=\exp(tL)$ from (78) maps forward-going data into forward-going data and backward-going data. So, we write

$$\tilde{V}(t)=\begin{pmatrix}\tilde{V}_{FF}(t) & \tilde{V}_{FB}(t)\\ \tilde{V}_{BF}(t) & \tilde{V}_{BB}(t)\end{pmatrix}\quad . \tag{87}$$

Here, $\tilde{V}_{FF}$ maps forward-going data into forward-going data, $\tilde{V}_{FB}$ maps forward-going data into backward-going data, $\tilde{V}_{BF}$ maps backward-going data into forward-going data and $\tilde{V}_{BB}$ maps backward-going data into backward-going data.

From $(81)_2$ and (86) we obtain for $\tilde{V}_{FF}$

$$\tilde{V}_{FF}(t)=T(t)-\int_0^t\int_0^{t_1} T(t-t_1)\gamma T'(t_1-t_2)\gamma T(t_2)\mathrm{d}t_1\mathrm{d}t_2+\ldots.. \tag{88}$$

The first term in (88) is simply translating a forward-going datum into a forward direction. The integrant $T(t-t_1)\gamma T'(t_1-t_2)\gamma T(t_2)$ translate a forward-going datum in the forward direction from time zero to time t_2 when it is reflected. On reflection it is multiplied by the local reflection coefficient γ, then translated backwards from time t_2 to time t_1, when it is reflected again, multiplied by γ and translated forwards from time t_1 to time t. So, the second term in (88) represents the contribution to the forward-going disturbance from all possible double reflections. The following terms in (88) consider third reflections and so on.

Knowing this, it is easy to write

$$\tilde{V}_{FB}(t)=\int_0^t T'(t-t_1)\gamma T(t_1)dt_1 - \\ -\int_0^t\int_0^{t_1}\int_0^{t_2} T'(t-t_1)\gamma T(t_1-t_2)\gamma T'(t_2-t_3)\gamma T(t_3)dt_1dt_2dt_3+..... \tag{89}$$

$$\tilde{V}_{BF}(t)=-\int_0^t T'(t-t_1)\gamma T(t_1)dt_1 - \\ -\int_0^t\int_0^{t_1}\int_0^{t_2} T(t-t_1)\gamma T'(t_1-t_2)\gamma T(t_2-t_3)\gamma T'(t_3)dt_1dt_2dt_3+..... \tag{90}$$

$$\tilde{V}_{BB}(t)=T'(t)-\int_0^t\int_0^{t_1} T'(t-t_1)\gamma T(t_1-t_2)\gamma T'(t_2)dt_1dt_2+..... \tag{91}$$

Now, from (78), (87) and (89)-(91) we have

$$V(t)=\begin{pmatrix} V_{11}(t) & V_{12}(t) \\ V_{21}(t) & V_{22}(t)\end{pmatrix}, \tag{92}$$

with

$$V_{11}(t)=\frac{1}{2}\sqrt{c}[V_{FF}(t)+V_{BB}(t)+V_{FB}(t)+V_{BF}(t)]\frac{1}{\sqrt{c}},$$

$$V_{12}(t)=\frac{1}{2}\sqrt{c}[V_{BB}(t)-V_{FF}(t)+V_{FB}(t)-V_{BF}(t)]\frac{1}{\sqrt{c}},$$

$$V_{21}(t)=\frac{1}{2}\sqrt{c}[V_{BB}(t)-V_{FF}(t)+V_{BF}(t)-V_{FB}(t)]\frac{1}{\sqrt{c}}, \tag{93}$$

$$V_{22}(t)=\frac{1}{2}\sqrt{c}[V_{FF}(t)+V_{BB}(t)-V_{FB}(t)-V_{BF}(t)]\frac{1}{\sqrt{c}}.$$

Taking account of the initial data (83) we have

$$y_t(u,t)=\frac{1}{2}\sqrt{c(u)}[V_{BB}(t)-V_{FF}(t)+V_{BF}(t)-V_{FB}(t)]\frac{1}{\sqrt{c(u)}}\phi'(u)+ \\ +\frac{1}{2}\sqrt{c(u)}[V_{BB}(t)+V_{FF}(t)-V_{BF}(t)-V_{FB}(t)]\frac{1}{\sqrt{c(u)}}\psi(u) \tag{94}$$

and

$$y(u,t)=\phi(u)+\frac{1}{2}\sqrt{c(u)}\int_0^t [V_{BB}(t)-V_{FF}(t)+V_{BF}(t)-V_{FB}(t)]\frac{1}{\sqrt{c(u)}}\phi'(u)dt_1+$$
$$+\frac{1}{2}\sqrt{c(u)}\int_0^t [V_{BB}(t)+V_{FF}(t)-V_{BF}(t)-V_{FB}(t)]\frac{1}{\sqrt{c(u)}}\psi(u)dt_1 \quad (95)$$

For $c'(u)=0$, we obtain

$$y(u,t)=\phi(u)+\frac{1}{2}\int_0^t [T'(t_1)-T(t_1)]\phi'(u)dt_1+\frac{1}{2}\int_0^t [T'(t_1)+T(t_1)]\psi(u)dt_1 \quad . \quad (96)$$

After integration by parts and a change of variable (96) yields to D'Alembert formula (84).
In conclusion, from (95) we see that the solution of

$$y_{tt}=A^2(x)y_{xx},$$

is expressed in term of the

$$\sqrt{c(u)}=\sqrt{A(x)}\ ,$$

where

$$x\to u(x)=\int_0^x \frac{dz}{A(z)},$$

and

$$A(x)=A(y_x(x)),$$

verifying the equation

$$A(x)=\frac{\lambda[e_3+(e_2-e_3)\mathrm{sn}^2(\sqrt{e_1-e_3}y_x+\delta')]}{1+\rho[e_3+(e_2-e_3)\mathrm{sn}^2(\sqrt{e_1-e_3}y_x+\delta')]}, \quad (97)$$

or

$$A(x)=\frac{\lambda[e_1+(e_1-e_3)\mathrm{sech}^2(\sqrt{e_1-e_3}y_x+\delta')]}{1+\rho[e_1+(e_1-e_3)\mathrm{sech}^2(\sqrt{e_1-e_3}y_x+\delta')]} \quad . \quad (98)$$

So, the solutions are expressed in term of cnoidal (or soliton) waves.

Therefore, we can emphasis that the collisions between such solutions have a cnoidal or soliton behavior. The applications of the theory presented in this paper can be found in [15, 16]. [17]

In [17], the effect of hysteresis on the wave propagation in DRIP media is discussed for a simple wave profile $f(x)=\mathrm{sech}\,x$. This is the case of interaction of two pulses having a soliton profile, travelling in a DRIP medium.

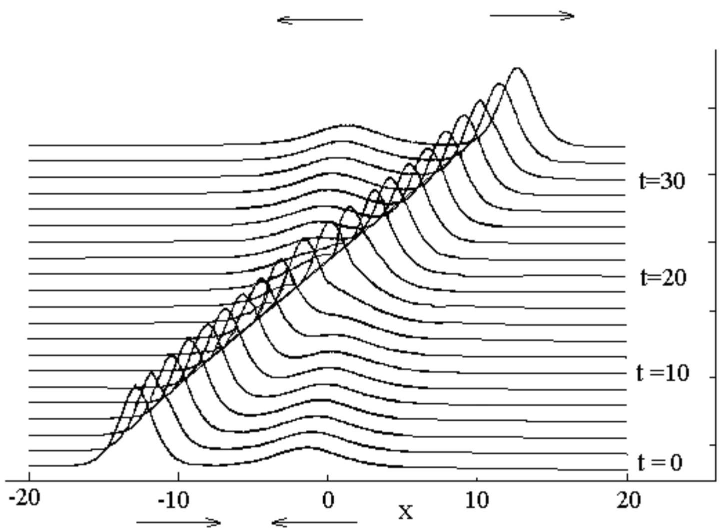

Fig.2. Profiles of waves without hysteresis against x for several t [17].

Fig. 2 illustrates the case of interactions without hysteresis in DRIP media. From this figure we see that, in contrast to the theory of solitons, these waves move in opposite directions, interact and emerge unaffected by the interaction. In the interaction area no coupling between waves is visible. This suggests that the waves may be regarded individually. Speaking from a physical viewpoint, this interaction requires that the energy of each field is carried individually without any transfer of energy between fields.

In fig. 3, the interaction region exhibits the coupling between waves. Therefore, the waves may not be regarded individually. The coupling of waves can be explained by the energy transfer between waves. This property may be of the transmitting medium rather than of the particular wave profiles [17].

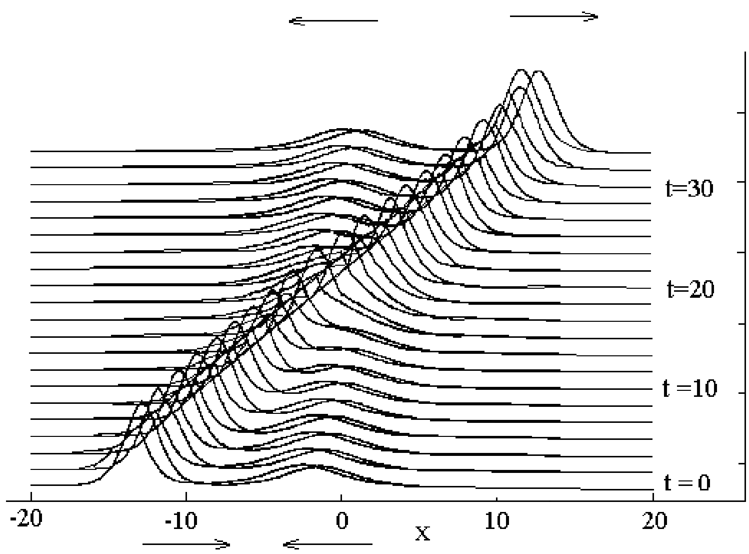

Fig. 3. Profiles of waves with hysteresis against x for several t [17].

6. **CONCLUSIONS**

The waves described by Seymour and Varley distort as they propagate, and are of arbitrary shape and amplitude. Since such media transmit waves that *do not remember the interaction process*, they are called *DRIP media*. Seymour and Varley [1] analyse DRIP Media whose responses are governed by the nonlinear nondispersive wave equation, in which any two pulses traveling in opposite directions interact nonlinearly for a finite time when they collide but then part unaffected by the interaction. When any two pulses are traveling in opposite directions meet and interact, they emerge from the interaction region unchanged by the interaction. This interaction is similar to those that occurs when two solitons collide. The main difference is that solitons are represented by waves of permanent form whose profiles are specific.

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