Romanian Journal of Mechanics
Volume 6, Issue 2/2021, pp 31 -45, ISSN 2537 - 5229, rjm.journals.srmta.ro

A TENSOR CLOSED-FORM SOLUTION OF TWO-BODY PROBLEM IN ROTATING REFERENCE FRAME

Daniel CONDURACHE*, Ana Maria COMANICI*

* "Gheorghe Asachi" Technical University of Iasi, Department of Theoretical Mechanics, Romania
Corresponding author: Daniel CONDURACHE, E-mail: daniel.condurache@tuiasi.ro

Abstract The paper presents a comprehensive analysis, together with the derivation of a closed-form solution to the two-body problem in rotating non-inertial reference frames. By using an efficient tensor mathematical instrument, which is closely related to the attitude kinematics methods, the motion in the rotating non-inertial reference frame is completely solved. The closed-form solutions for the motion in the non-inertial frame, the motion of the mass center, and the relative motion are presented. Dynamical characteristics similar to linear momentum, angular momentum, and total energy are introduced.

Key words: two-body problem, non-inertial reference frame, closed-form solution

1. INTRODUCTION

This paper studies the two-body problem in non-inertial rotating reference frames, offering its solution in the general case.

The mathematical model for the two-problem in non-inertial rotating reference frame is represented by the initial value problems:

$$m_1\ddot{\mathbf{r}}_1 + 2m_1\boldsymbol{\omega}\times\dot{\mathbf{r}}_1 + m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1) + m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}_1 = \mathbf{F}_{12},\quad \begin{cases}\mathbf{r}_1(t_0)=\mathbf{r}_1^0\\ \dot{\mathbf{r}}_1(t_0)=\mathbf{v}_1^0\end{cases}; \tag{1}$$

$$m_2\ddot{\mathbf{r}}_2 + 2m_2\boldsymbol{\omega}\times\dot{\mathbf{r}}_2 + m_2\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_2) + m_2\dot{\boldsymbol{\omega}}\times\mathbf{r}_2 = \mathbf{F}_{21},\quad \begin{cases}\mathbf{r}_2(t_0)=\mathbf{r}_2^0\\ \dot{\mathbf{r}}_2(t_0)=\mathbf{v}_2^0\end{cases}; \tag{2}$$

where $\mathbf{r}_k$ denotes the position vector, $\mathbf{v}_k$ the velocity vector of the particle P_k , $k=\overline{1,2}$, related to the reference frame where the motion takes place, m_k the masses of the two particles and $\boldsymbol{\omega}$ the angular velocity of the non-inertial reference frame where the motion takes place. The vectorial map $\boldsymbol{\omega}$ is supposed to be differentiable. $\mathbf{F}_{ij}$ denotes the force that actions on the particle P_i due to the interaction with the particle P_j , $i\neq j,\ i,j=\overline{1,2}$. From the reciprocal interaction's principle:

$$\mathbf{F}_{12}+\mathbf{F}_{21}=\mathbf{0} \tag{3}$$

$$\mathbf{F}_{12} \times (\mathbf{r}_1 - \mathbf{r}_2) = \mathbf{0} \tag{4}$$

We will prove that the two-body problem in non-inertial reference frames may be solved exactly like in the inertial case, by solving two particle problems:

1. The motion of the mass center.
2. The relative motion of one particle related to another, for example the motion of particle P_2 related to P_1 (Fig. 1). It is described by the vectorial map $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, where $\mathbf{r}_2$ is the solution to the initial value problem (2) and $\mathbf{r}_1$ to the initial value problem (1).

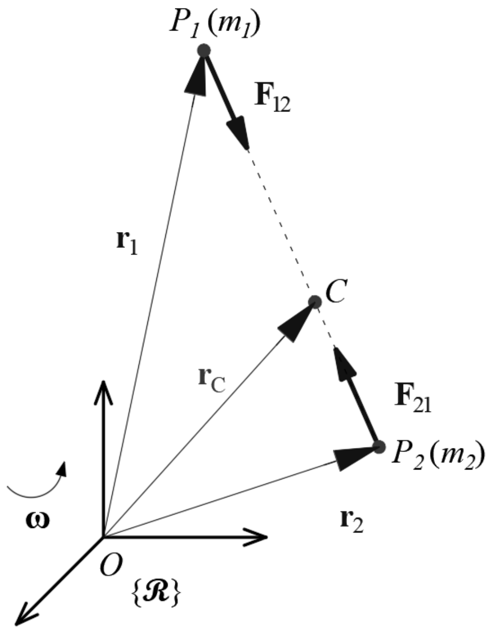

Fig. 1 The two-body problem in a non-inertial reference frame

Section 2 introduces the tensorial instrument used in this paper. It is based on proper orthogonal tensorial maps and skew-symmetric tensorial maps. A differential vectorial operator is introduced, and it is useful in studying an arbitrary motion related to a non-inertial reference frame in general.

Using the adequate tensorial instrument introduced in Section 2, Section 3 offers:

- An explicit solution for the center of mass motion is given, in the general case.
- A representation theorem of the relative motion is offered; it reduces the problem to the study of one particle motion in a central positional force field.
- New prime integrals are obtained, replicas to angular momentum and energy conservation.

A new prime integral for the system motion is offered, replicas to impulse, angular momentum and kinetic energy theorems. The motion of the system related to the mass center reference frame is also studied.

Let us remark that a comprehensive study on the two-body problem in non-inertial reference frames misses from the classical textbooks ([1], [2], [4], [5], [6], [7], [8]). The reason could be the very unusual form of the prime integrals.

2. TENSORIAL CONSIDERATIONS

This section introduces the main mathematical instruments used in this paper. A tensorial map and a vectorial differential operator will be defined.

We will denote:

- $\mathbf{V}_3$ the three-dimensional vectorial space.
- $\mathbf{V}_3^{\mathbb{R}}$ the set of the maps defined on the positive real semiaxis with values in $\mathbf{V}_3$.
- $\mathbf{SO}_3$ the special orthogonal group of second order tensors.
- $\mathbf{SO}_3^{\mathbb{R}}$ the set of the maps defined on the positive real semiaxis with values in $\mathbf{SO}_3$
- $\mathbf{so}_3$ the Lie-algebra of skew-symmetric second order tensors.
- $\mathbf{so}_3^{\mathbb{R}}$ the set of maps defined on the positive real semiaxis with values in $\mathbf{so}_3$.

2.1 A Tensorial Operator

The rotation motion with arbitrary angular velocity $\boldsymbol{\omega}$ will be related to orthogonal tensorial maps.

Lemma 1: *The initial value problem*:

$$\begin{aligned} \dot{\mathbf{Q}} &= \mathbf{Q}\tilde{\boldsymbol{\omega}} \\ \mathbf{Q}(t_0) &= \mathbf{I}_3 \end{aligned} \tag{5}$$

with $t_0 \geq 0$ *has a unique solution* $\mathbf{Q} \in \mathbf{SO}_3^{\mathbb{R}}$ *for any continuous map* $\tilde{\boldsymbol{\omega}} \in \mathbf{so}_3^{\mathbb{R}}$.

Proof. It is obvious that (5) has a unique solution $\mathbf{Q} = \mathbf{Q}(t)$. It only remains to prove that this solution $\mathbf{Q}$ is in $\mathbf{SO}_3^{\mathbb{R}}$, meaning that $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_3$ and $\det(\mathbf{Q}) = 1$. We have that $\frac{d}{dt}\left(\mathbf{Q}\mathbf{Q}^T\right) = \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T = \mathbf{Q}\tilde{\boldsymbol{\omega}}\mathbf{Q}^T - \mathbf{Q}\tilde{\boldsymbol{\omega}}\mathbf{Q}^T = \mathbf{0}_3$, so $\mathbf{Q}\mathbf{Q}^T$ is a constant differentiable function that satisfies $\left(\mathbf{Q}\mathbf{Q}^T\right)(t_0) = \mathbf{I}_3$. Then, $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}_3$. Using that $\det(\mathbf{Q})$ is also a continuous function which satisfies $\det(\mathbf{Q}) \in \{-1, 1\}$ and $\det(\mathbf{Q}(t_0)) = \det \mathbf{I}_3 = 1$, it comes that $\det \mathbf{Q} = 1$. So $\mathbf{Q} \in \mathbf{SO}_3^{\mathbb{R}}$.□

Remark 2: Lemma 1 is the famous Darboux problem (see [3]): finding the rotation tensor when knowing the instantaneous angular velocity. The link between the rotation tensorial map (also called orthogonal tensorial map) and the skew-symmetric tensor associated to the angular velocity vector is given by the initial value problem (5).

The solution to the initial value problem (5) will be denoted $\mathbf{F}_{\boldsymbol{\omega}}$. The next result presents the properties of this tensorial orthogonal map.

Lemma 3: *The map* $\mathbf{F}_{\boldsymbol{\omega}}$ *satisfies:*

1. $\mathbf{F}_{\boldsymbol{\omega}}$ *is invertible.*
2. $\mathbf{F}_{\boldsymbol{\omega}}\mathbf{u} \cdot \mathbf{F}_{\boldsymbol{\omega}}\mathbf{v} = \mathbf{u} \cdot \mathbf{v},\ (\forall)\mathbf{u}, \mathbf{v} \in \mathbf{V}_3^{\mathbb{R}}$;
3. $\left|\mathbf{F}_{\boldsymbol{\omega}}\mathbf{u}\right| = |\mathbf{u}|,\ (\forall)\mathbf{u} \in \mathbf{V}_3^{\mathbb{R}}$;
4. $\mathbf{F}_{\boldsymbol{\omega}}(\mathbf{u} \times \mathbf{v}) = \mathbf{F}_{\boldsymbol{\omega}}\mathbf{u} \times \mathbf{F}_{\boldsymbol{\omega}}\mathbf{v},\ (\forall)\mathbf{u}, \mathbf{v} \in \mathbf{V}_3^{\mathbb{R}}$;
5. $\frac{d}{dt}\mathbf{F}_{\boldsymbol{\omega}}\mathbf{u} = \mathbf{F}_{\boldsymbol{\omega}}(\dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{u}),\ (\forall)\mathbf{u} \in \mathbf{V}_3^{\mathbb{R}}$, *differentiable.*
6. $\frac{d^2}{dt^2}\mathbf{F}_{\boldsymbol{\omega}}\mathbf{u} = \mathbf{F}_{\boldsymbol{\omega}}\left(\ddot{\mathbf{u}} + 2\boldsymbol{\omega} \times \dot{\mathbf{u}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{u}) + \dot{\boldsymbol{\omega}} \times \mathbf{u}\right),\ (\forall)\mathbf{u} \in \mathbf{V}_3^{\mathbb{R}}$, *two times differentiable.*

The proof to **Lemma 3** involves only elementary computations and it will be skipped.

2.2 Comments and Remarks

1. We will denote:

$$\left(\mathbf{F}_{\boldsymbol{\omega}}\right)^{-1} \overset{not}{=} \mathbf{R}_{-\boldsymbol{\omega}} \tag{6}$$

As $\mathbf{F}_{\omega}$ is the solution to the initial value problem (1), it results that $\mathbf{R}_{-\boldsymbol{\omega}}$ is the angular velocity $-\boldsymbol{\omega}$ rotation tensor and it is the solution to the initial value problem:

$$\begin{aligned} \dot{\mathbf{Q}} + \tilde{\boldsymbol{\omega}}\mathbf{Q} &= \mathbf{0} \\ \mathbf{Q}(t_0) &= \mathbf{I}_3. \end{aligned} \tag{7}$$

2. In case $\boldsymbol{\omega}$ has fixed direction, $\boldsymbol{\omega} = \omega\mathbf{u}$, with $\mathbf{u}$ constant unit vector and $\omega : \mathbb{R} \to \mathbb{R}$, as $\tilde{\boldsymbol{\omega}}(t_1)\tilde{\boldsymbol{\omega}}(t_2) = \tilde{\boldsymbol{\omega}}(t_2)\tilde{\boldsymbol{\omega}}(t_1)$,$(\forall)t_{1,2} \in \mathbb{R}$, (see [3]), then $\mathbf{R}_{-\boldsymbol{\omega}(t)}$ has the explicit form:

$$\mathbf{R}_{-\boldsymbol{\omega}(t)} = \exp\left(-\int_{t_0}^{t} \tilde{\boldsymbol{\omega}}(\xi)\, d\xi\right) = \mathbf{I}_3 - \frac{\sin\varphi(t)}{\boldsymbol{\omega}}\tilde{\omega} + \frac{1-\cos\varphi(t)}{\boldsymbol{\omega}^2}\tilde{\boldsymbol{\omega}}^2, \tag{8}$$

where $\varphi(t) = \int_{t_0}^{t} \omega(\xi)\, d\xi$.

In case $\boldsymbol{\omega}$ is constant, $\mathbf{R}_{-\boldsymbol{\omega}(t)}$ has the explicit form:

$$\mathbf{R}_{-\boldsymbol{\omega}(t)} = \exp\left[-(t-t_0)\tilde{\boldsymbol{\omega}}\right] = \mathbf{I}_3 - \frac{\sin\left[\boldsymbol{\omega}(t-t_0)\right]}{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}} + \frac{1-\cos\left[\boldsymbol{\omega}(t-t_0)\right]}{\boldsymbol{\omega}^2}\tilde{\boldsymbol{\omega}}^2. \tag{9}$$

Remark 4: Eqs. (8) and (9) give the explicit solution to Darboux problem in case vector $\boldsymbol{\omega}$ has fixed direction, respectively it is constant.

2.3 A Vectorial Operator

We introduce a differential operator which is related to the angular velocity $\boldsymbol{\omega}$ of the reference frame to whom an arbitrary vector is related. It is a derivation-like operator, and its use will be revealed further.

We define operator $(\ )' : \mathbf{V}_3^{\mathbb{R}} \to \mathbf{V}_3^{\mathbb{R}}$ by:

$$(\ )' = (\dot{\ }) + \boldsymbol{\omega} \times (\ ). \tag{10}$$

For any arbitrary vectorial map $\mathbf{u} : \mathbb{R} \to \mathbf{V}_3^{\mathbb{R}}$, it will hold:

$$\mathbf{u}' = \dot{\mathbf{u}} + \boldsymbol{\omega} \times \mathbf{u}. \tag{11}$$

The next result presents the properties of this operator, together with the link between $(\ )'$ and $\mathbf{F}_{\boldsymbol{\omega}}$.

Lemma 5: *The following affirmations hold true:*

1. $\boldsymbol{\omega}' = \dot{\boldsymbol{\omega}}$;
2. $(\mathbf{u}+\mathbf{v})' = \mathbf{u}' + \mathbf{v}'$, $(\forall)\mathbf{u},\mathbf{v} \in C^2\left(\mathbf{V}_3^{\mathbb{R}}\right)$;
3. *$(\lambda\mathbf{u})' = \dot{\lambda}\mathbf{u} + \lambda\mathbf{u}'$, $(\forall)\mathbf{u} \in C^2\left(\mathbf{V}_3^{\mathbb{R}}\right)$,$(\forall)\lambda : \mathbb{R} \to \mathbb{R}$, differentiable;*
4. $(\mathbf{u}\times\mathbf{v})' = \mathbf{u}'\times\mathbf{v} + \mathbf{u}\times\mathbf{v}'$, $(\forall)\mathbf{u},\mathbf{v} \in C^2\left(\mathbf{V}_3^{\mathbb{R}}\right)$;
5. $\mathbf{u}'\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v}' = \dot{\mathbf{u}}\cdot\mathbf{v} + \mathbf{u}\cdot\dot{\mathbf{v}} = \frac{d}{dt}(\mathbf{u}\cdot\mathbf{v})$, $(\forall)\mathbf{u},\mathbf{v} \in C^2\left(\mathbf{V}_3^{\mathbb{R}}\right)$;
6. $\mathbf{u}'' = \ddot{\mathbf{u}} + 2\boldsymbol{\omega}\times\dot{\mathbf{u}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{u}) + \dot{\boldsymbol{\omega}}\times\mathbf{u}$, $(\forall)\mathbf{u} \in C^2\left(\mathbf{V}_3^{\mathbb{R}}\right)$;
7. $\frac{d}{dt}(\mathbf{F}_{\boldsymbol{\omega}}\mathbf{u}) = \mathbf{F}_{\boldsymbol{\omega}}(\mathbf{u}')$, $(\forall)\mathbf{u} \in C^2\left(\mathbf{V}_3^{\mathbb{R}}\right)$.

The proof to **Lemma 5** involves only elementary computations and it will also be skipped.

An interesting property of operator $(\ )'$ is revealed in next **Lemma**.

Lemma 6: *Let* $\mathbf{u}:\mathbb{R}_+ \to \mathbf{V}_3^{\mathbb{R}}$ *be a differential vectorial valued map such as:*

$$\mathbf{u}' = \mathbf{0}\ ,\ \mathbf{u}(t_0) = \mathbf{u}_0 . \tag{12}$$

Then:

$$\mathbf{u} = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{u}_0 \tag{13}$$

where $(\mathbf{R}_{-\boldsymbol{\omega}})^T$ *is the solution to the initial value problem* (5).

Proof. From $\frac{d}{dt}(\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{u}_0) = \dot{\mathbf{R}}_{-\boldsymbol{\omega}}\mathbf{u}_0 = -\tilde{\boldsymbol{\omega}}\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{u}_0$, it results $\frac{d}{dt}(\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{u}_0) + \boldsymbol{\omega}\times(\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{u}_0) = \mathbf{0}$. The unique solution of the initial value problem (12) is $\mathbf{u} = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{u}_0$.□

Remark 7: From **Lemma 6** it results that if a vectorial map $\mathbf{u}:\mathbb{R}_+ \to \mathbf{V}_3^{\mathbb{R}}$ satisfies $\mathbf{u}' = \mathbf{0}$, then vector $\mathbf{u}$ is the rotation with angular velocity $-\boldsymbol{\omega}$ of a constant vector $\mathbf{u}_0 = \mathbf{u}(t_0)$. It will be useful in giving a geometrical interpretation for the prime integrals that occur in the two-body problem in non-inertial reference frames.

3. THE STUDY OF THE TWO BODY PROBLEM IN ROTATING NON-INERTIAL REFERENCE FRAMES

This is the main section of this paper: the core of the theoretical study we offer here. The motion of the mass center, the relative motion of a particle related to another, its prime integrals, system-related prime integrals and the motion related to the center mass non-inertial reference frame are studied. The essential result is **Theorem 10**, which relates the inertial and the non-inertial two-body problems via an orthogonal proper tensorial map.

3.1 The Mass Center Motion

This section offers an exact vectorial solution in the mass center motion problem. We denote by:

$$\mathbf{r}_C = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \tag{14}$$

the position vector of the mass center of the two particles related to the non-inertial reference frame where the motion takes place. It results that

$$\dot{\mathbf{r}}_C = \frac{m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2}{m_1 + m_2}. \tag{15}$$

By summarizing eqs (1) and (2) and taking into account conditions (3) and (4), it results that the initial value problem that describes its motion is:

$$\ddot{\mathbf{r}}_C + 2\boldsymbol{\omega}\times\dot{\mathbf{r}}_C + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_C) + \dot{\boldsymbol{\omega}}\times\mathbf{r}_C = \mathbf{0},$$
$$\begin{cases} \mathbf{r}_C(t_0) = \dfrac{m_1\mathbf{r}_1^0 + m_2\mathbf{r}_2^0}{m_1 + m_2} \overset{not}{=} \mathbf{r}_C^0 \\ \dot{\mathbf{r}}_C(t_0) = \dfrac{m_1\mathbf{v}_1^0 + m_2\mathbf{v}_2^0}{m_1 + m_2} \overset{not}{=} \mathbf{v}_C^0 \end{cases}. \tag{16}$$

Theorem 8: *The solution to the initial value problem (16) is*

$$\mathbf{r}_C(t) = \mathbf{R}_{-\boldsymbol{\omega}}\left[\mathbf{r}_C^0 + \left(\mathbf{v}_C^0 + \boldsymbol{\omega}^0 \times \mathbf{r}_C^0\right)(t - t_0)\right] \ , \ t \in [t_0, +\infty), \tag{17}$$

where $\boldsymbol{\omega}^0 = \boldsymbol{\omega}(t_0)$.

Proof. Applying operator $\mathbf{F}_{\boldsymbol{\omega}}$ to the initial value problem (17) and using **Lemma 3**, it results that the initial value problem (17) becomes:

$$\ddot{\boldsymbol{\rho}} = \mathbf{0} \ ,$$
$$\begin{cases} \boldsymbol{\rho}(t_0) = \mathbf{r}_C^0 \\ \dot{\boldsymbol{\rho}}(t_0) = \mathbf{v}_C^0 + \boldsymbol{\omega}^0 \times \mathbf{r}_C^0 \end{cases} , \tag{18}$$

where $\boldsymbol{\rho} \overset{not}{=} \mathbf{F}_{\boldsymbol{\omega}}\mathbf{r}$. The solution to the initial value problem (18) is

$$\boldsymbol{\rho}(t) = \mathbf{r}_C^0 + \left(\mathbf{v}_C^0 + \boldsymbol{\omega}^0 \times \mathbf{r}_C^0\right)(t - t_0) \ , \ t \in [t_0, +\infty). \tag{19}$$

Taking into account that $\left(\mathbf{F}_{\boldsymbol{\omega}}\right)^{-1} = \mathbf{R}_{-\boldsymbol{\omega}}$, the conclusion of the theorem is proved. □

Remark 9:

1. As $\left|\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{u}\right| = |\mathbf{u}|$, $(\forall)\mathbf{u} \in \left(\mathbf{V}_3^{\mathbb{R}}\right)$, it results that at any moment of time the mass center is on a variable sphere that has a linear-increasing radius: $\mathbf{r}_C^0 + \left(\mathbf{v}_C^0 + \boldsymbol{\omega}^0 \times \mathbf{r}_C^0\right)(t - t_0)$.

2. In case $\mathbf{v}_C^0 + \boldsymbol{\omega}^0 \times \mathbf{r}_C^0 = \mathbf{0}$, the motion of the mass center takes place on a sphere with radius $\mathbf{r}_C^0$.

3. In case vector $\boldsymbol{\omega}$ has fixed direction, the law of motion of the mass center may be written explicitly, taking into account the expression (8) of the tensorial map $\mathbf{R}_{-\boldsymbol{\omega}}$.

4. The hodograph of the vectorial map that models the mass center motion is a curve that is situated on a ruled surface generated by the rotation with angular velocity $-\boldsymbol{\omega}$ of the straight line:

$$\mathbf{r} = \mathbf{r}_C^0 + \left(\mathbf{v}_C^0 + \boldsymbol{\omega}^0 \times \mathbf{r}_C^0\right)(t - t_0) \ , \ t \in [t_0, +\infty). \tag{20}$$

Considering the non-inertial reference frame having only a rotation motion, the straight line defined in eq (20) has a fixed point, so it generates a conical surface. We may then state that the motion of the mass center may be decomposed into two:

- a rectilinear uniform motion with velocity $\mathbf{v}_C^0 + \boldsymbol{\omega}^0 \times \mathbf{r}_C^0$
- a rotation with angular velocity $-\boldsymbol{\omega}$ of the straight line where the rectilinear motion takes place.

The trajectory generated by these two independent motions is situated on a conical surface.

3.2 The Relative Motion

The motion of particle $P_2(m_2)$ related to particle $P_1(m_1)$ is described by the vectorial map:

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1, \tag{21}$$

where $\mathbf{r}_1$, $\mathbf{r}_2$ are the solutions to thew initial value problems (1), respectively (2). Dividing eq (1) by m_1, eq (2) by m_2, and using conditions (3), (4), it results that vector $\mathbf{r}$ satisfies:

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega}\times\dot{\mathbf{r}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r} = \frac{\mathbf{F}_{21}}{m}, \tag{22}$$

where:

$$m = \frac{m_1 m_2}{m_1 + m_2} \tag{23}$$

denotes the reduced mass of the system. Vector $\mathbf{F}_{21}$ represents a central isotropical force and it may be written:

$$\mathbf{F}_{21} = f(r)\frac{\mathbf{r}}{r} \tag{24}$$

where $f:[t_0,+\infty)\to\mathbb{R}$ is a scalar map. It results that the relative motion of $P_1(m_1)$ related to $P_2(m_2)$ is described by the initial value problem:

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega}\times\dot{\mathbf{r}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r} = \frac{f(r)}{m}\frac{\mathbf{r}}{r},$$
$$\begin{cases} \mathbf{r}(t_0) = \mathbf{r}_2^0 - \mathbf{r}_1^0 \overset{not}{=} \mathbf{r}^0 \\ \dot{\mathbf{r}}(t_0) = \mathbf{v}_2^0 - \mathbf{v}_1^0 \overset{not}{=} \mathbf{v}^0 \end{cases}. \tag{25}$$

3.2.1 A Relative Motion Representation Theorem

The initial value problem (25) describes the motion of a particle having the mass equal with the reduced mass of the system related to the non-inertial reference frame the two-body problem is related to. The solution to the initial value problem (25) is obtained by using the next **Theorem**. This is a result that may be applied in the general motion related to a non-inertial reference frame. In fact, it relates the inertial and the non-inertial motion via proper orthogonal tensorial maps.

Theorem 10: *The solution to the initial value problem (25) is obtained by applying operator* $\mathbf{R}_{-\boldsymbol{\omega}}$ *to the solution to the initial value problem:*

$$\ddot{\mathbf{r}} = \frac{f(r)}{m}\frac{\mathbf{r}}{r},$$
$$\begin{cases} \mathbf{r}(t_0) = \mathbf{r}^0 \\ \dot{\mathbf{r}}(t_0) = \mathbf{v}^0 + \boldsymbol{\omega}^0\times\mathbf{r}^0 \end{cases}, \tag{26}$$

where $\boldsymbol{\omega}^0 = \boldsymbol{\omega}(t_0)$

Proof. Eq (25) may be written using the previous considerations:

$$\mathbf{r}'' = \frac{f(r)}{m}\frac{\mathbf{r}}{r}. \tag{27}$$

Applying operator $\mathbf{F}_{\boldsymbol{\omega}}$ to eq (27) and using that

$$\frac{d^2}{dt^2}(\mathbf{F}_{\boldsymbol{\omega}}\mathbf{r}) = \mathbf{F}_{\boldsymbol{\omega}}(\mathbf{r}''), \tag{28}$$

it results:

$$\mathbf{F}_{\boldsymbol{\omega}}\left(\mathbf{r}''\right)=\frac{f\left(r\right)}{m}\frac{\mathbf{F}_{\boldsymbol{\omega}}\mathbf{r}}{\left|\mathbf{F}_{\boldsymbol{\omega}}\mathbf{r}\right|}. \tag{29}$$

Denoting $\boldsymbol{\rho}=\mathbf{F}_{\boldsymbol{\omega}}\mathbf{r}$, the link between the two differential equations becomes evident. The initial conditions for the new initial value problem become:

$$\boldsymbol{\rho}\left(t_0\right)=\mathbf{F}_{\boldsymbol{\omega}}\mathbf{r}\Big|_{t=t_0}=\mathbf{r}^0, \tag{30}$$

$$\dot{\boldsymbol{\rho}}\left(t_0\right)=\frac{d}{dt}\left(\mathbf{F}_{\boldsymbol{\omega}}\mathbf{r}\right)\Big|_{t=t_0}=\mathbf{v}^0+\boldsymbol{\omega}^0\times\mathbf{r} \tag{31}$$

Using eq (6), it results that applying operator $\mathbf{R}_{-\boldsymbol{\omega}}=\left(\mathbf{F}_{\boldsymbol{\omega}}\right)^{-1}$ to the solution to the initial value problem (26), the solution to the initial value problem (25) is obtained:

$$\mathbf{r}=\mathbf{R}_{-\boldsymbol{\omega}}\boldsymbol{\rho}\ . \tag{32}$$

The proof is finalized. □

Remark 11: Theorem 10 offers a simple way of solving the two-body problem in non-inertial reference frames:

- the initial value problem with modified initial conditions (26) is solved.
- the tensorial operator $\mathbf{R}_{-\boldsymbol{\omega}}$ is applied to the solution to the initial value problem (26). The solution to the initial value problem (25) is obtained.

3.2.2 The Prime Integrals of the Relative Motion

This section studies the prime integrals of the initial value problem (25). They are deduced using the tensorial instruments introduced in Section 2. We use the following denotation:

$$\boldsymbol{\Omega}_0\overset{not}{=}\mathbf{r}^0\times\left(\mathbf{v}^0+\boldsymbol{\omega}^0\times\mathbf{r}^0\right) \tag{33}$$

Theorem 12: *The initial value problem (25) has the prime integrals:*

$$\mathbf{r}\times\left(\dot{\mathbf{r}}+\boldsymbol{\omega}\times\mathbf{r}\right)=\mathbf{R}_{-\boldsymbol{\omega}}\boldsymbol{\Omega}_0\overset{not}{=}\boldsymbol{\Omega} \tag{34}$$

(Replica to angular momentum conservation);

$$\frac{m\dot{\mathbf{r}}^2}{2}+m\left(\boldsymbol{\omega},\mathbf{r},\dot{\mathbf{r}}\right)+\frac{m}{2}\left(\boldsymbol{\omega}\times\mathbf{r}\right)^2-\int f\left(r\right)dr=\text{constant}\overset{not}{=}h \tag{35}$$

(Replica to energy conservation).

Proof. Using operator $\left(\ \right)'$ introduced in Section 2, it results:

$$\left(\mathbf{r}\times\mathbf{r}'\right)'=\mathbf{r}\times\mathbf{r}''=\mathbf{0} \tag{36}$$

and from **Lemma 6** it results:

$$\mathbf{r}\times\mathbf{r}'=\mathbf{R}_{-\boldsymbol{\omega}}\left[\left(\mathbf{r}\times\mathbf{r}'\right)\Big|_{t=t_0}\right]=\mathbf{R}_{-\boldsymbol{\omega}}\boldsymbol{\Omega}_0\,. \tag{37}$$

The second prime integral is deduced by derivation. □

Remarks:

1. The prime integral (34) shows that the hodograph of the vectorial map $\mathbf{\Omega} = \mathbf{r} \times (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r})$ is a spherical curve, as it is the rotation with angular velocity $-\boldsymbol{\omega}$ of a constant vector $\mathbf{\Omega}_0$. In case vector $\boldsymbol{\omega}$ has fixed direction, the hodograph of $\mathbf{\Omega}$ is a circular section, as it is the rotation of a constant vector around a fixed axis. Vector $\mathbf{\Omega}$ "sweeps" the lateral surface of a right circular cone with angular velocity $-\boldsymbol{\omega}$. As it results from eq (8), its explicit expression is:

$$\mathbf{\Omega} = \frac{\mathbf{\Omega}_0 \cdot \boldsymbol{\omega}}{\boldsymbol{\omega}^2}\boldsymbol{\omega} - \frac{\sin\left[\varphi(t) - \varphi(t_0)\right]}{\boldsymbol{\omega}}\boldsymbol{\omega} \times \mathbf{\Omega}_0 - \frac{\cos\left[\varphi(t) - \varphi(t_0)\right]}{\boldsymbol{\omega}^2}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{\Omega}_0) \tag{38}$$

where $\varphi(t) = \int_{t_0}^{t} \omega(\tau) d\tau$.

2. The second prime integral (35) has energetic signification. It emphasizes the existence of a potential energy role map:

$$V(t, \mathbf{r}, \dot{\mathbf{r}}) = m(\boldsymbol{\omega}, \mathbf{r}, \dot{\mathbf{r}}) + \frac{m}{2}(\boldsymbol{\omega} \times \mathbf{r})^2 - \int f(r) dr . \tag{39}$$

The prime integral (35) may be rewritten:

$$\frac{m}{2}\dot{\mathbf{r}}^2 + V(t, \mathbf{r}, \dot{\mathbf{r}}) = \text{constant} . \tag{40}$$

If $\boldsymbol{\omega}$ has a constant direction, $\mathbf{u}$ then $\boldsymbol{\omega}(t) = \omega(t)\mathbf{u}$. As $\mathbf{r} \times (\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{\Omega}_0$, by dot-multiplying this relation with $\boldsymbol{\omega}$, we get:

$$(\boldsymbol{\omega}, \mathbf{r}, \dot{\mathbf{r}}) + (\boldsymbol{\omega} \times \mathbf{r})^2 = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{\Omega}_0 \cdot \boldsymbol{\omega} = \mathbf{\Omega}_0 \cdot (\mathbf{R}_{-\boldsymbol{\omega}})^{-1} \boldsymbol{\omega} = \mathbf{\Omega}_0 \cdot \boldsymbol{\omega} . \tag{41}$$

(If $\boldsymbol{\omega}$ has constant direction, then $(\mathbf{R}_{-\boldsymbol{\omega}})^{-1} = \mathbf{R}_{\boldsymbol{\omega}}$ and $\mathbf{R}_{\boldsymbol{\omega}}\boldsymbol{\omega} = \boldsymbol{\omega}$). So, $(\boldsymbol{\omega}, \mathbf{r}, \dot{\mathbf{r}}) + (\boldsymbol{\omega} \times \mathbf{r})^2 = \mathbf{\Omega}_0 \cdot \boldsymbol{\omega}$ and in correlation with (39) we get:

$$V = m\left[\mathbf{\Omega}_0 \cdot \boldsymbol{\omega} - \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{r})^2\right] - \int f(r) dr . \tag{42}$$

In this case $V = V(t, \mathbf{r})$.

If $\boldsymbol{\omega}$ is a constant vector, from (42) it results: $V = V(\mathbf{r})$ (the classical potential energy).

3.2.3 The Laws of Motion in the Non-Inertial Reference Frame

Knowing the solutions of the uni-particle problems described by eqs (16) and (25) solves the laws of motion problem for the particles $P_1(m_1)$, $P_2(m_2)$ in the non-inertial reference frame where the motion takes place. From relations (14) and (21) it results:

$$\mathbf{r}_1 = \mathbf{r}_C - \frac{m_2}{m_1 + m_2}\mathbf{r} , \tag{43}$$

$$\mathbf{r}_2 = \mathbf{r}_C + \frac{m_1}{m_1 + m_2}\mathbf{r} . \tag{44}$$

The vectorial map $\mathbf{r}_C$ is the solution of the initial value problem (16) and has the expression from eq (17):

$$\mathbf{r}_C\left(t\right)=\mathbf{R}_{-\boldsymbol{\omega}}\left[\mathbf{r}_C^0+\left(\mathbf{v}_C^0+\boldsymbol{\omega}^0\times\mathbf{r}_C^0\right)\left(t-t_0\right)\right]\ ,\ t\in\left[t_0,+\infty\right). \tag{45}$$

The vectorial map $\mathbf{r}$ is the solution to the initial value problem (25), that is determined by applying tensor $\mathbf{R}_{-\boldsymbol{\omega}}$ to the solution to the initial value problem (26).

3.3 The Prime Integrals of the Two-Body System

This Section studies the global mechanical characteristics of the whole system of particles. By introducing replicas for the classic mechanical characteristics of a system of particles (impulse, angular momentum, kinetic energy), interesting prime integrals are deduced.

It is known that the classic (inertial) mechanical characteristics of a two-particle system are:

$$\mathbf{P}=\sum_{k=1}^{2}m_k\dot{\mathbf{r}}_k \tag{46}$$

(Impulse);

$$\mathbf{K}=\sum_{k=1}^{2}m_k\left(\mathbf{r}_k\times\dot{\mathbf{r}}_k\right) \tag{47}$$

(Angular momentum);

$$E_{kin}=\frac{1}{2}\sum_{k=1}^{2}m_k\dot{\mathbf{r}}_k^2 \tag{48}$$

(Kinetic energy).

In a non-inertial reference frame, the conservation laws of these quantities do not apply. By defining:

$$\mathbf{H}=\sum_{k=1}^{2}m_k\left(\dot{\mathbf{r}}_k+\boldsymbol{\omega}\times\mathbf{r}_k\right) \tag{49}$$

(Generalized impulse);

$$\mathbf{L}=\sum_{k=1}^{2}m_k\left[\mathbf{r}_k\times\left(\dot{\mathbf{r}}_k+\boldsymbol{\omega}\times\mathbf{r}_k\right)\right] \tag{50}$$

(Generalized angular momentum);

$$T=\frac{1}{2}\sum_{k=1}^{2}m_k\left(\dot{\mathbf{r}}_k+\boldsymbol{\omega}\times\mathbf{r}_k\right)^2 \tag{51}$$

(Generalized potential energy), the following result may be stated:

Theorem 13: *The following affirmations hold true:*

$$\dot{\mathbf{H}}+\boldsymbol{\omega}\times\mathbf{H}=\mathbf{0} \tag{52}$$

(Replica to impulse conservation law);

$$\dot{\mathbf{L}}+\boldsymbol{\omega}\times\mathbf{L}=\mathbf{0} \tag{53}$$

(Replica to angular momentum conservation law);

$$T - \int f(r)dr = \text{constant} \tag{54}$$

(Replica to kinetic energy conservation law). $-\int f(r)dr$ represents e interaction potential energy of the system.

The proof of **Theorem 13** is made by elementary computations, and it will be skipped.

Remark 14:

1. Eq (52) may be rewritten using operator $(\ )'$

$$\mathbf{H}' = \mathbf{0}. \tag{55}$$

Using **Lemma 6**, it results that the replica to impulse conservation law may be written:

$$\mathbf{H} = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{H}_0 = \mathbf{R}_{-\boldsymbol{\omega}}\left[\sum_{k=1}^{2} m_k\left(\mathbf{v}_k^0 + \boldsymbol{\omega}^0 \times \mathbf{r}_k^0\right)\right]. \tag{56}$$

2. Eq (53) may be rewritten using operator $(\ )'$

$$\mathbf{L}' = \mathbf{0}. \tag{57}$$

Using **Lemma 6**, it results that the replica to angular momentum conservation law may be written:

$$\mathbf{L} = \mathbf{R}_{-\boldsymbol{\omega}}\mathbf{L}_0 = \mathbf{R}_{-\boldsymbol{\omega}}\left\{\sum_{k=1}^{2} m_k\left[\mathbf{r}_k^0 \times \left(\mathbf{v}_k^0 + \boldsymbol{\omega}^0 \times \mathbf{r}_k^0\right)\right]\right\}. \tag{58}$$

It is natural to give now the replicas to the general laws of conservation in the classic case. From definitions (49), (50), (51), from relations:

$$\mathbf{H} = \mathbf{P} + (m_1 + m_2)\boldsymbol{\omega} \times \mathbf{r}_C \tag{59}$$

$$\mathbf{L} = \mathbf{K} + \mathbf{I}_0\boldsymbol{\omega} \tag{60}$$

$$T = E_{kin} + V_C \tag{61}$$

with:

$$\mathbf{I}_0 = \sum_{k=1}^{2} m_k \tilde{\mathbf{r}}_k \tilde{\mathbf{r}}_k^T. \tag{62}$$

$$V_C = \sum_{k=1}^{2}\left[m_k\left(\boldsymbol{\omega}, \mathbf{r}_k, \dot{\mathbf{r}}_k\right) + m_k \frac{\left(\boldsymbol{\omega} \times \mathbf{r}_k\right)^2}{2}\right] \tag{63}$$

($\mathbf{r}_C$ denotes the position vector of the mess center related to the non-inertial frame, $\mathbf{I}_0$ is the inertia tensor related to the non-inertial frame), we state:

Corollary 15: *The prime integrals of the two-particle system are:*

$$\left|\mathbf{P} + (m_1 + m_2)\boldsymbol{\omega} \times \mathbf{r}_C\right| = \text{constant}; \tag{64}$$

$$\left|\mathbf{K} + \sum_{k=1}^{2} \mathbf{r}_k \times \left(\boldsymbol{\omega} \times \mathbf{r}_k\right)\right| = \text{constant}; \tag{65}$$

$$E_{kin}+\sum_{k=1}^{2}\left[m_k\left(\boldsymbol{\omega},\mathbf{r}_k,\dot{\mathbf{r}}_k\right)+m_k\frac{\left(\boldsymbol{\omega}\times\mathbf{r}_k\right)^2}{2}\right]-\int f\left(r\right)dr=\text{constant}. \qquad (66)$$

If vector $\boldsymbol{\omega}$ has fixed direction, the vectorial prime integrals (52) and (53) have explicit formulations:

$$\mathbf{H}=\frac{\mathbf{H}_0\cdot\boldsymbol{\omega}}{\boldsymbol{\omega}^2}\boldsymbol{\omega}-\frac{\sin\left[\varphi(t)-\varphi(t_0)\right]}{\boldsymbol{\omega}}\boldsymbol{\omega}\times\mathbf{H}_0-\frac{\cos\left[\varphi(t)-\varphi(t_0)\right]}{\boldsymbol{\omega}^2}\boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{H}_0\right); \qquad (67)$$

$$\mathbf{L}=\frac{\mathbf{L}_0\cdot\boldsymbol{\omega}}{\boldsymbol{\omega}^2}\boldsymbol{\omega}-\frac{\sin\left[\varphi(t)-\varphi(t_0)\right]}{\boldsymbol{\omega}}\boldsymbol{\omega}\times\mathbf{L}_0-\frac{\cos\left[\varphi(t)-\varphi(t_0)\right]}{\boldsymbol{\omega}^2}\boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{L}_0\right), \qquad (68)$$

where:

$$\mathbf{H}_0=\mathbf{H}\left(t_0\right); \qquad (69)$$

$$\mathbf{L}_0=\mathbf{L}\left(t_0\right). \qquad (70)$$

The hodographs of vectors $\mathbf{H}$ and $\mathbf{L}$ are circular sections. $\mathbf{H}$ and $\mathbf{L}$ "sweep" the lateral surface of a right circular cone with angular velocity $-\boldsymbol{\omega}$.

3.4 The Motion Related to the Mass Center Reference Frame

The motion of the particles $P_1\left(m_1\right)$, $P_2\left(m_2\right)$ related to the non-inertial reference frame of the mass center may be completely described if the solutions to the initial value problems (16) and (25). We denote:

$$\mathbf{r}_1^*=\mathbf{r}_1-\mathbf{r}_C; \qquad (71)$$

$$\mathbf{r}_2^*=\mathbf{r}_2-\mathbf{r}_C; \qquad (72)$$

From relations:

$$m_1\mathbf{r}_1^*+m_2\mathbf{r}_2^*=\mathbf{0}; \qquad (73)$$

$$\mathbf{r}_2^*-\mathbf{r}_1^*=\mathbf{r}; \qquad (74)$$

it results:

$$\mathbf{r}_1^*=-\frac{m_2}{m_1+m_2}\mathbf{r}; \qquad (75)$$

$$\mathbf{r}_2^*=\frac{m_1}{m_1+m_2}\mathbf{r}. \qquad (76)$$

Relations (75) and (76) represent the laws of motion of particles $P_1\left(m_1\right)$, $P_2\left(m_2\right)$ related to the non-inertial reference frame of the mass center. The initial value problems that describe their motion in this frame are ($k=\overline{1,2}$):

$$\ddot{\mathbf{r}}_k^* + 2\boldsymbol{\omega}\times\dot{\mathbf{r}}_k^* + \boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{r}_k^*\right) + \dot{\boldsymbol{\omega}}\times\mathbf{r}_k^* = \frac{f\left(r_k^*\right)}{m_k}\frac{\mathbf{r}_k^*}{r_k^*} ,$$
$$\begin{cases}\mathbf{r}_k^*\left(t_0\right) = \mathbf{r}_k^0 - \mathbf{r}_C^0 \\ \dot{\mathbf{r}}_k^*\left(t_0\right) = \mathbf{v}_k^0 - \mathbf{v}_C^0\end{cases} . \tag{77}$$

The mechanical global characteristics of the system in this reference frame are:

$$\mathbf{H}^C = \sum_{k=1}^{2} m_k \left(\dot{\mathbf{r}}_k^* + \boldsymbol{\omega}\times\mathbf{r}_k^*\right) \tag{78}$$

(Generalized impulse);

$$\mathbf{L}^C = \sum_{k=1}^{2} m_k \left[\mathbf{r}_k^* \times\left(\dot{\mathbf{r}}_k^* + \boldsymbol{\omega}\times\mathbf{r}_k^*\right)\right] \tag{79}$$

(Generalized angular momentum);

$$T^C = \frac{1}{2}\sum_{k=1}^{2} m_k \left(\dot{\mathbf{r}}_k^* + \boldsymbol{\omega}\times\mathbf{r}_k^*\right)^2 \tag{80}$$

(Generalized kinetic energy).

From relations (78)-(80), taking into account (75), (76), it results:

$$\mathbf{H}^C = \mathbf{0} \tag{81}$$

$$\mathbf{L}^C = m\left[\mathbf{r}\times\left(\dot{\mathbf{r}} + \boldsymbol{\omega}\times\mathbf{r}\right)\right] \tag{82}$$

$$T^C = \frac{1}{2} m \left(\dot{\mathbf{r}} + \boldsymbol{\omega}\times\mathbf{r}\right)^2 \tag{83}$$

where $m = \dfrac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of the system and $\mathbf{r}$ is the solution to the initial value problem (25).

It results the vectorial prime integral of the motion:

$$\mathbf{L}^C = \mathbf{R}_{-\boldsymbol{\omega}} \mathbf{L}_0^C \tag{84}$$

that shows that the hodograph of vector $\mathbf{L}$ is a spherical curve. If $\boldsymbol{\omega}$ has constant direction, this hodograph is a circular section; vector $\mathbf{L}$ sweeps the surface of a right circular cone with angular velocity $-\boldsymbol{\omega}$. From (82) results:

$$\mathbf{r}\cdot\mathbf{L}^C = 0 , \tag{85}$$

so a geometrical visualization of the motion may be given: the two particles are situated at any moment of time in a variable plane that is normal on vector $\mathbf{L}$. It also results that the trajectories of the two particles are spatial homothetical to C curves, as follows from:

$$\mathbf{r}_1^* = -\frac{m_2}{m_1}\mathbf{r}_2^* . \tag{86}$$

The homothety ratio is $-\dfrac{m_2}{m_1}$.

Remarks:

1.The trajectories are plane curves if and only if vector $\mathbf{L}^C$ has fixed direction, as follows from (85). The following result may be stated:

Lemma 16: *The trajectories in the two-body problem in the mass center non-inertial reference frame are planar if and only the conditions below are satisfied:*

(i) *Vector* $\boldsymbol{\omega}$ *has fixed direction,* $\boldsymbol{\omega} = \omega(t)\mathbf{u}$, *with* $\omega:[t_0,\infty)\to\mathbb{R}$ *and* $\mathbf{u}$ *constant unit vector.*

(ii) $\mathbf{u}\times\mathbf{L}_0^C = \mathbf{0}$, *where* $\mathbf{L}_0^C = \mathbf{L}^C(t_0)$.

Proof. "$\Rightarrow$" If the trajectory is a plane curve, it results vector $\mathbf{L}^C$ is constant, as it is normal on this plane and has constant magnitude. It results $\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{L}_0^C = \text{constant}$, so $\frac{d}{dt}\left(\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{L}_0^C\right) = \mathbf{0}$. It results $\dot{\mathbf{R}}_{-\boldsymbol{\omega}}\mathbf{L}_0^C = 0$, so $-\tilde{\boldsymbol{\omega}}\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{L}_0^C = 0$. Further: $\boldsymbol{\omega}\times\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{L}_0^C = 0$. It results

$$\boldsymbol{\omega}\times\mathbf{L}^C = \mathbf{0}. \tag{87}$$

As vector $\mathbf{L}^C$ is constant, it results vector $\boldsymbol{\omega}$ has fixed direction, that of $\mathbf{L}^C$. As $\boldsymbol{\omega} = \omega(t)\mathbf{u}$, from (87) it results $\mathbf{u}\times\mathbf{L}_0^C = \mathbf{0}$.

"$\Leftarrow$" If conditions (i) and (ii) are satisfied, it results:

From (i): $\mathbf{R}_{-\boldsymbol{\omega}}\boldsymbol{\omega} = \boldsymbol{\omega}$ and $\left(\mathbf{R}_{-\boldsymbol{\omega}}\right)^{-1} = \mathbf{R}_{\boldsymbol{\omega}}$.

From (ii): $\mathbf{0} = \mathbf{u}\times\mathbf{L}_0^C = \left(\mathbf{R}_{\boldsymbol{\omega}}\mathbf{u}\right)\times\mathbf{L}_0^C = \left(\mathbf{R}_{\boldsymbol{\omega}}\mathbf{u}\right)\times\left[\left(\mathbf{R}_{-\boldsymbol{\omega}}\right)^{-1}\mathbf{L}^C\right] = \left(\mathbf{R}_{\boldsymbol{\omega}}\mathbf{u}\right)\times\left(\mathbf{R}_{\boldsymbol{\omega}}\mathbf{L}^C\right)$. It results further: $\mathbf{0} = \left(\mathbf{R}_{\boldsymbol{\omega}}\mathbf{u}\right)\times\left(\mathbf{R}_{\boldsymbol{\omega}}\mathbf{L}^C\right) = \mathbf{R}_{\boldsymbol{\omega}}\left(\mathbf{u}\times\mathbf{L}^C\right)$, so $\mathbf{u}\times\mathbf{L}^C = \mathbf{0}$. It results vector $\mathbf{L}^C$ has constant direction. As it has constant magnitude, too, it results it is constant, so the trajectory is a planar curve. The proof is finalized. □

2. If the interaction forces between the particles depend only on the relative distance r, a scalar prime integral with energetic significance may be deduced:

$$T^C - \int f(r)dr = \text{constant}. \tag{88}$$

The equivalent form of eq (88) is:

$$\frac{m}{2}\dot{\mathbf{r}}^2 + m(\boldsymbol{\omega},\mathbf{r},\dot{\mathbf{r}}) + \frac{m}{2}(\boldsymbol{\omega}\times\mathbf{r})^2 - \int f(r)dr = \text{constant}. \tag{89}$$

Knowing that:

$$E_{kin} = \frac{m}{2}\dot{\mathbf{r}}^2 \tag{90}$$

is the kinetic energy of the system related to the non-inertial frame of the mass center and denoting:

$$V(t,\mathbf{r},\dot{\mathbf{r}}) = m(\boldsymbol{\omega},\mathbf{r},\dot{\mathbf{r}}) + \frac{m}{2}(\boldsymbol{\omega}\times\mathbf{r})^2 - \int f(r)dr, \tag{91}$$

the generalized potential energy, relation (89) becomes:

$$E_{kin} + V(t,\mathbf{r},\dot{\mathbf{r}}) = \text{constant}. \tag{92}$$

Eq (92) shows that in the two-body problem in the arbitrary rotating non-inertial reference frame of the mass center (with angular velocity $\boldsymbol{\omega}$), there exists the prime integral (92) with the generalized potential energy introduced in eq (91).

In case the angular velocity $\boldsymbol{\omega}$ has constant direction, the prime integral (92) becomes:

$$E_{kin} + V\left(t, \mathbf{r}\right) = \text{constant}\,, \tag{93}$$

$$V\left(t, \mathbf{r}\right) = m\left[\boldsymbol{\Omega}_0 \cdot \boldsymbol{\omega} - \frac{1}{2}\left(\boldsymbol{\omega} \times \mathbf{r}\right)^2\right] - \int f\left(r\right)dr, \qquad \boldsymbol{\Omega}_0 = \mathbf{r}^0 \times \left(\mathbf{v}^0 + \boldsymbol{\omega}^0 \times \mathbf{r}^0\right) \tag{94}$$

and in case $\boldsymbol{\omega}$ is constant:

$$E_{kin} + V\left(\mathbf{r}\right) = \text{constant}\,, \tag{95}$$

$$V\left(\mathbf{r}\right) = -\frac{m}{2}\left(\boldsymbol{\omega} \times \mathbf{r}\right)^2 - \int f\left(r\right)dr\,. \tag{96}$$

4. CONCLUSIONS

The study of the two-body problem in a rotating non-inertial reference frame was approached comprehensively, and its closed-form solution was determined. The motion of the center of mass, the relative motion of the two bodies in the non-inertial rotating reference frame were approached by using the same tensor instrument. The results were presented in vectorial coordinate-free expressions. First integrals, equivalent to the classical dynamical characteristics of the motion (linear momentum, angular momentum, total energy), were determined. The present approach generalizes the classic two body-problem. Future works will study the same context for concrete forms of the interaction force (including the classic gravitational case), as well as several situations of other central forces.

5. REFERENCES

1. APPELL, P., *Traite de mecanique rationelle* (5 vol.), Gauthier-Villars, Paris, 1926.
2. ARNOLD, V.I., *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1989 (Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein).
3. DARBOUX, G., *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal*, Tome 1, Cap II, Gautiers-Villars, Paris, 1887.
4. GANTMACHER, F.R., *The Theory of Matrices*, New York, Chelsea, 1959.
5. LEVI-CIVITA., T., *Sur la régulisation du problème des trois corps*, Acta Math., **42**, 1920.
6. MARSDEN, J.E., *Lectures on Mechanics*, Cambridge University Press, 1992.
7. MARSDEN, J.E., RATIU T., *Introduction to Mechanics and Symmetry. A Basic Exposition on Classical Mechanical Systems*, Number 17 in Texts in Appl. Math. Springer Verlag, 1999.
8. CONDURACHE, D., *The Two Body Problem in Non-Inertial Reference Frames*, The 2nd International Symposium of Theoretical and Applied Mechanics "Dimitrie I. Mangeron", Iasi, Romania, 2005 (keynote presentation).

Received November 27, 2021