Romanian Journal of Mechanics
Volume 3, Issue 1/2018, pp 39-48, ISSN 2537 - 5229, *rjm.journals.srmta.ro*

ON THE CHETAEV NONHOLONOMIC SYSTEMS

Rodica IOAN [*,**] , Valeria MOȘNEGUȚU [**]

[*] University Spiru Haret, Bucharest
[**] Institute of Solid Mechanics, Romanian Academy, Bucharest

Corresponding author: Rodica IOAN, E-mail: rodicaioan08@yahoo.com

Abstract In this paper, the Chetaev nonholonomic constraints of the form $f_k(t,q,\dot{q})=0$, $k=1,2,...,p$, are treated as initial conditions attached to the Appell equations of motion of a dynamical system. The rolling of a uniform disk without sliding on a horizontal plane is solving by using the cnoidal method.

Key words: Nonholonomic systems, kinematic constraints, Chetaev nonholonomic constraints, Appell equations, rolling disk, cnoidal method.

1. INTRODUCTION

The constraint represents a restriction on positions and velocities of a dynamical system. A system of points with fixed distances between points (rigid body), motion of a rolling body with no slip condition, are examples of such constraints. The first constraint is a holonomic constraint, while in the second example, the no-slip condition is a linear nonholonomic constraint.

The nonintegrable kinematics constraints, nonreducible to holonomic constrains are called *nonholonomic constraints*. A review on nonholonomic systems is found in [1]. Other works focussing on this issue are [2-11]. Nonholonomic systems typically arise when constraints on velocity are imposed, such as the constraint that the bodies roll without slipping on a surface. Cars, bicycles, unicycles - anything with rolling wheels - are all examples of nonholonomic systems.

Nonholonomic constrains date back to the time of Euler, Lagrange and d'Alembert. The geometry of nonholonomic systems shares its mathematical foundations with geometric control theory, control problems and sub-Riemannian geometry.

A common example of nonholonomic systems refers to a dog pursuing a man - the man is walking from the origin O of the coordinate system Oxy and moves along the y -axis with velocity c. His dog starts at the same moment from the point (x_0, y_0), $x_0 \geq 0$, $y_0 \neq 0$, and runs so its velocity at each moment is the line that binds its instantaneous position and the instantaneous position of the man. The problem is to find the trajectory of the dog [7].

Another example is the problem of Leibnitz (1689), i.e. to find a curve along which a particle moves in a homogeneous gravitational field with constant velocity. The solution is the paracentric isochrone curve and it was given by Jacob Bernoulli in 1694 [12, 13]. A similar example refers to motion of a particle in a homogeneous gravitational field with initial velocity starting from a given point and subjected to condition of a constant velocity [14]. A generalization of the last problem leads to the motion of a particle in a homogeneous gravitational field subjected to a nonlinear constraint [15]. The rolling disc on a horizontal plane, a homogeneous ball on a rotating surface are also example of nonholonomic systems [11,16].

Interesting applications of the nonholonomic systems are the robotic locomotion [17-22] and control of undulatory robotic locomotion [23-26] such as undulatory locomotion in the snake-like or worm-like motion. In these problems, the constructing of integrators is an open problem [23, 24]. A number of recent papers [25-28] is focused on the discrete version of the Lagrange-d'Alembert principle.

In this paper, the systems subjected to p Chetaev nonholonomic constraints are studied. The ideal bilateral Chetaev nonholonomic constraints

$$f_k(t,q,\dot{q}) = 0\,,\;\; k = 1,2,...,p\,, \tag{1}$$

are treated as initial conditions attached to the Appell motion equations of the dynamical system [29]. In (1), $q = (q_1, q_2, ..., q_n)$ are n generalized coordinates which define the position of the dynamical system. The Appell's equation of motion is described by Paul Emile Appel [30,31] and Josiah Willard Gibbs [32]. The Appell's equations are more convenient in solving the nonholonomic systems problems due to the fact that Appell's formulation is an application of Gauss principle of least constraint. The rolling of a uniform disk without sliding on the horizontal surface is solved next.

2. THE ROLLING DISK

The nonholonomic constraints are introduced in the motion Appell equations by using constaints multipliers. The approach is applying to the rocking, rolling with not sliding motion.

Firstly, we give an example concerning a particle of mass m which moves in 3D space. The Appell equations of motion are written as

$$\begin{aligned} m\ddot{q}_1 &= 2\lambda\dot{q}_1\,,\\ m\ddot{q}_2 &= 2\lambda\dot{q}_2\,,\\ m\ddot{q}_3 &= -mg - 2\lambda\dot{q}_3\,, \end{aligned} \tag{2}$$

where (q_1, q_2, q_3) define the position of the particle. The Chetaev nonholonomic constraint is written as

$$f = \dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 = 0\,, \tag{3}$$

From (2) and (3) we obtain

$$\lambda = -\frac{mg}{4\dot{q}_3}\,. \tag{4}$$

By substituting (4) in (2) we get

$$\begin{aligned} m\ddot{q}_1 &= -\frac{mg}{2\dot{q}_3}\dot{q}_1\,,\\ m\ddot{q}_2 &= -\frac{mg}{2\dot{q}_3}\dot{q}_2\,, \end{aligned} \tag{5}$$

$$m\ddot{q}_3 = -\frac{mg}{2}.$$

By applying the Lie symmetry of the Appell equations (2), some conserved quantities with physical meaning are deduced. These conserved quantities can be also found by using Newtonian mechanics, or by using the analytical mechanics methods.

The general form of the Appell equations of motion of a dynamical system are written as

$$\frac{\partial S}{\partial \ddot{q}_s} = Q_s + \lambda_k \frac{\partial f_k}{\partial \dot{q}_s}, \quad k = 1,2,...,p, \quad s = 1,2,...,n, \tag{6}$$

where $S(t,q,\dot{q},\ddot{q})$ is the energy of the system, $Q_s(t,q,\dot{q})$, $s = 1,2,...,n$,are the generalized forces, and $\lambda_k(t,q,\dot{q})$, $k = 1,2,...,p$, are the constraint multipliers. The term $\lambda_k \frac{\partial f_k}{\partial \dot{q}_s}$ represents the generalized constraint forces. Unknowns $\lambda_k(t,q,\dot{q})$ are determined from (1) and (6).

The solution of (6) gives the motion of the nonholonomic system with initial conditions given by (1).

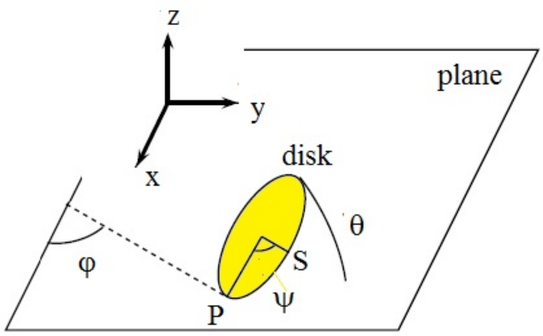

Fig.1. Rolling disk.

Consider now the problem of rolling disk without sliding on a horizontal plane Oxy [33]. The position of the disk is given by the coordinates (x, y) of the contact point P between the disk and the surface, the rotation angle ψ between P and an arbitrary point S on the disk in its motion, the angle φ between the tangent to the disk at P and the Ox axis, and the inclination angle θ between the disk and the surface (Fig. 1).

This problem is solved in [34] by using the symmetry group $E(2) \times S^1$ and a global gyroscopic stabilization principle. $E(2)$ is the Euclidean motion group of the plane, and S^1 is the group of internal symmetries of the disk. The symmetry simplifies the motion equations of motion.

The global gyroscopic stabilization principle, i.e. the relative equilibria are stable (elliptic) if their energy is larger than a fixed number. For exceeding values of the parameters, the disk falls flat for a certain time. A surprising result of this analysis is the existence of a universal constant change in the angle of the point of contact.

In this paper, the motion of the rolling disk is investigated by applying the cnoidal method [35].

The system of coordinates attached to the uniform disk is displayed in Fig. 2.

In addition to the 2D coordinates (x, z) with axis z vertically and axis x horizontally, we introduce the unit vectors (α, β, γ) related to the disk.

Axis α lies along the symmetry axis of the disk with the sense chosen so that the component ω_1 of the angular velocity vector ω of the disk with respect to this axis is positive, axis β is defined as $\beta = \gamma \times 1$ i.e. β lies in the direction of the velocity of the point of contact A (green), and axis γ is directed from the center of the disk O to A with the horizontal surface and makes the angle θ, $0 \le \theta \le \pi$, to that plane. The distance from O to A is a.

The motion of A is instantaneously in a circle of radius r.

The horizontal axis is $r = \beta \times z$. The distance from the axis of this motion to O is b. The horizontal axis is $r = \beta \times z$.

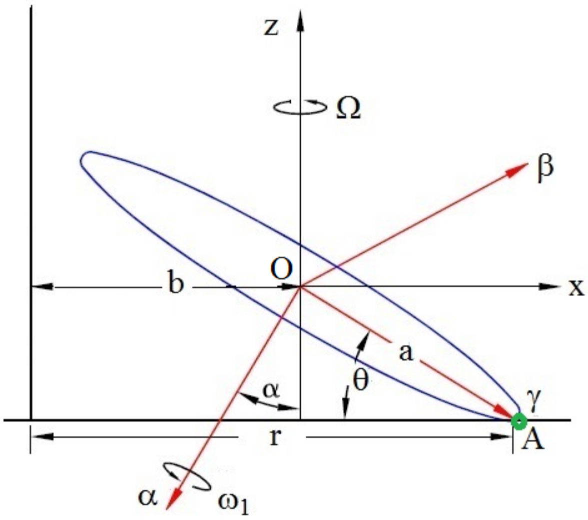

Fig. 2. System of coordinates.

The (α, β, γ) components of the equations of motion of the uniform disk are

$$\frac{3}{2}\dot{\omega}_1 + \dot{\theta}\Omega \sin\theta = 0,$$

$$\frac{1}{4}\Omega^2 \sin\theta\cos\theta + \frac{3}{2}\omega_1\Omega\sin\theta - \frac{5}{4}\ddot{\theta} = \frac{g}{a}\cos\theta, \tag{7}$$

$$\dot{\Omega}\sin\theta + 2\dot{\theta}\Omega\cos\theta + 2\omega_1\dot{\theta} = 0.$$

In this paper, the disk rolls without slipping relates the velocity of O to the angular velocity vector ω of the disk, i.e. the disk is subjected to the Chetaev nonholonomic constraint. In particular, the instantaneous velocity of A with the horizontal plane is zero, $v_O = v_A + \omega \times a\gamma = 0$.

The solutions of (7), $s(t) = \{\theta, \omega_1, \Omega\}$ are found as a sum of linear and nonlinear superposition of cnoidal vibrations, respectively [35]

$$s_i(t) = s_i^{lin}(\eta) + s_i^{int}(\eta), \quad i = 1,2,3, \tag{8}$$

$$s_i^{lin} = \sum_{l=1}^{n} \overline{\alpha}_l \mathrm{cn}^2[\overline{\omega}_l t; \overline{m}_l], \tag{9}$$

$$s_i^{\mathrm{int}} = \frac{\sum_{k=0}^{n} \overline{\beta}_k \mathrm{cn}^2[\eta; \overline{m}_k]}{1 + \sum_{k=0}^{n} \overline{\lambda}_k \mathrm{cn}^2[\eta; \overline{m}_k]}, \tag{10}$$

for $\eta = -\omega t$.

By applying the cnoidal method, we also can find more conserved quantities with physical meaning [36-40]. Ddetails can be found in [41-43].

3. RESULTS

Our results show that the rolling is stable only if

$$\omega_1^2 > \frac{g}{3a} \quad . \tag{11}$$

When the angular velocity about the vertical is $\Omega > \sqrt{\frac{4g}{a}}$ and $\omega_1 = \frac{b\Omega}{a} \neq 0$, the rotating disk is rising, that is, the plane of the disc may rise first towards the vertical, and then falling towards the horizontal. Fig.3 shows few snapshots of the trajectories of O nearly the disk falls flat. The trajectory of A before the disk falls are shown in Fig. 4. The results of Figs. 3 and 4 are analogous and correspond to those of Cushman [34] using the symmetry group $E(2)\times S^1$ and a global gyroscopic stabilization principle. The phase portraits $(\theta,\dot{\theta})$, $(\Omega,\dot{\Omega})$ and $(\omega_1,\dot{\omega}_1)$ nearly the disk falls flat are displayed in Fig.5.

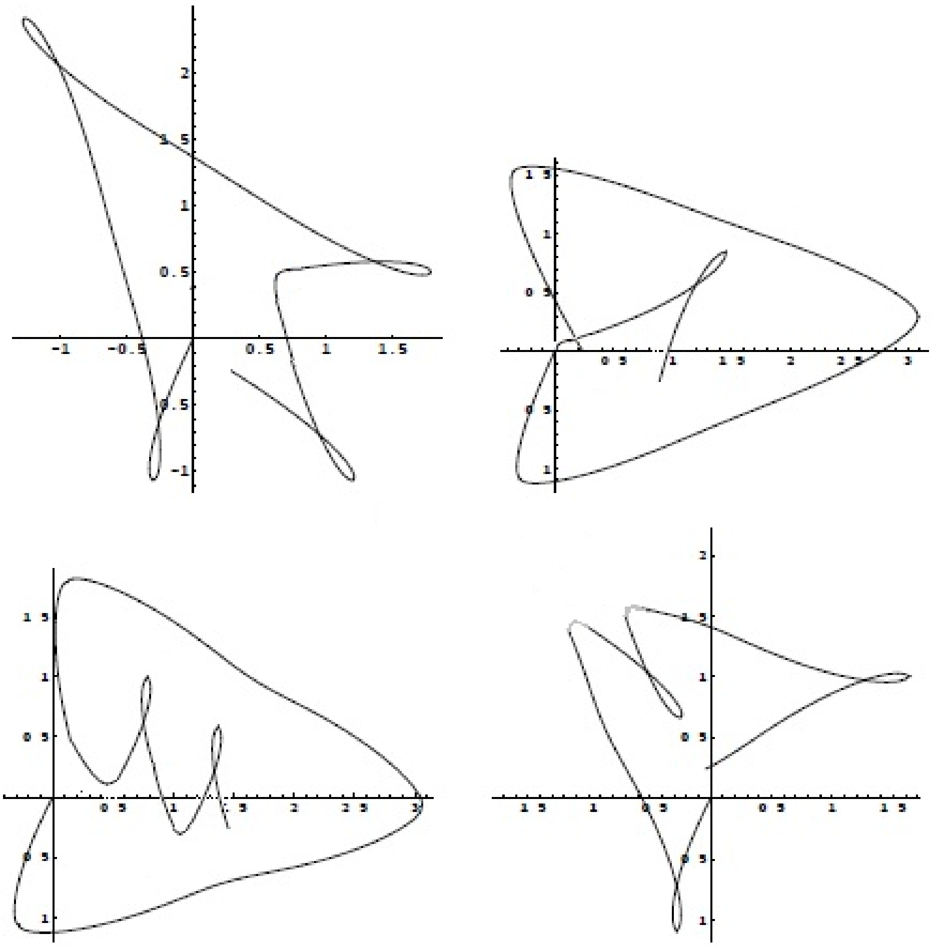

Fig. 3. Snapshots of the trajectories of O nearly the disk falls flat.

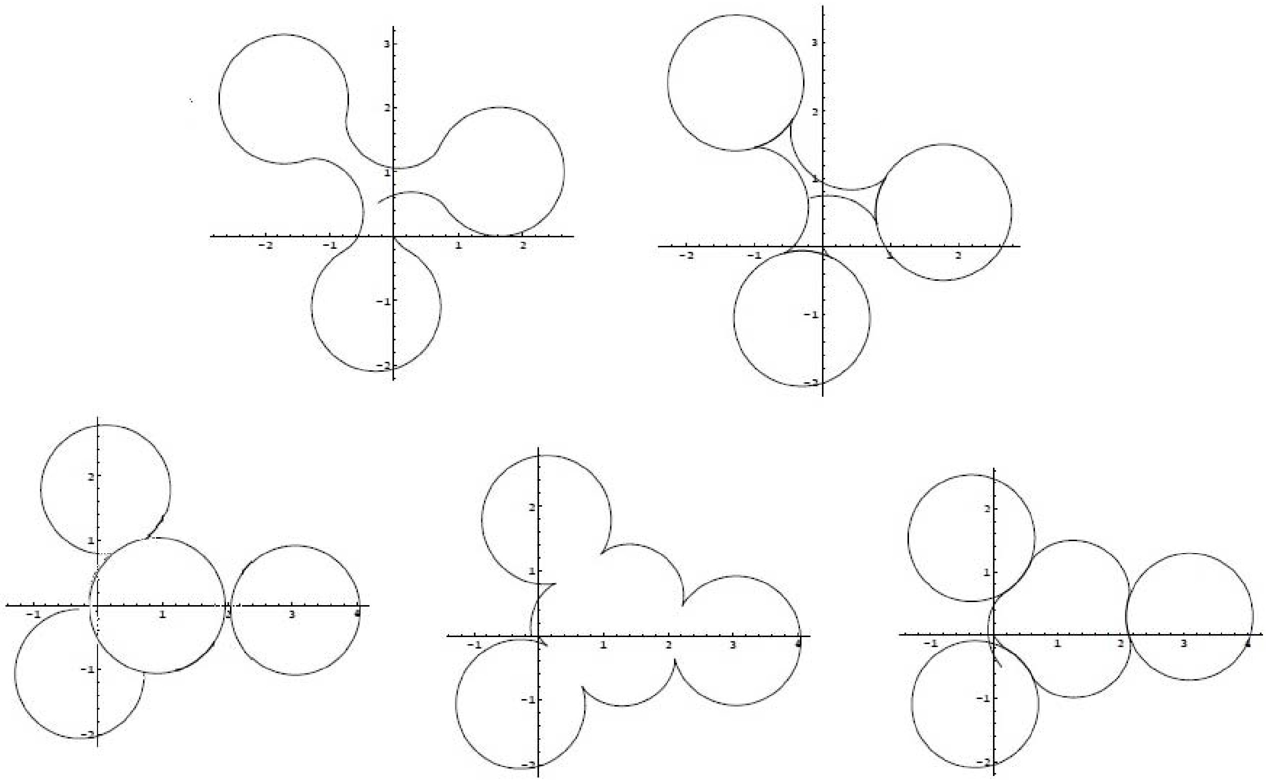

Fig. 4. Snapshots of the trajectories of *A* nearly the disk falls flat.

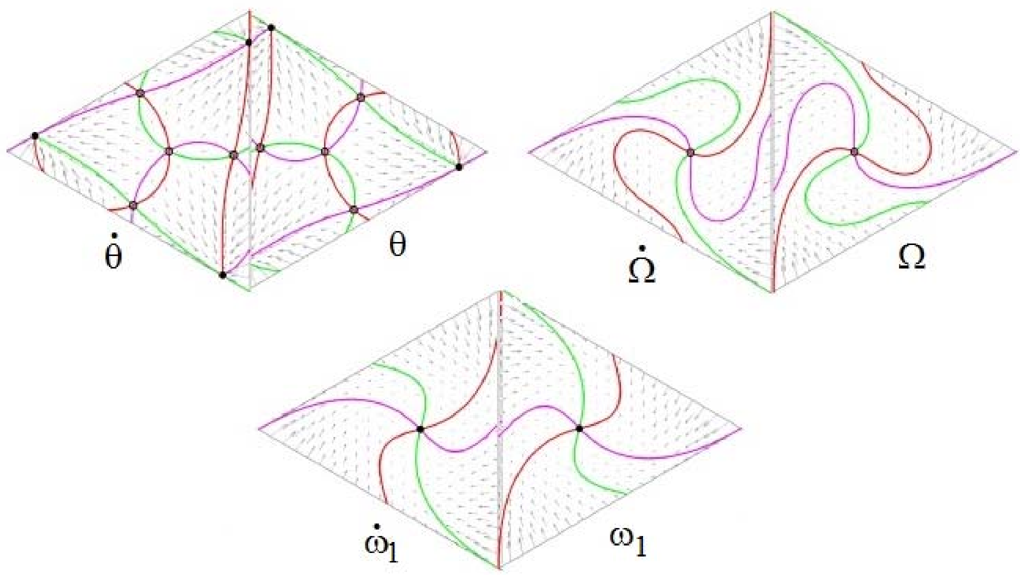

Fig. 5. Phase portraits $(\theta,\dot{\theta})$, $(\Omega,\dot{\Omega})$ and $(\omega_1,\dot{\omega}_1)$ nearly the disk falls flat.

The tendency to chaos is describes by attractors. An attractor is a set of numerical values toward which a dynamical system tends to evolve, for a wide variety of initial conditions. The Chetaev nonholonomic constraints are treated in this paper as initial conditions. These initial conditions depend on the motion of the system, according to (1) and (3). So, the rolling disk is strongly sensible to Chetaev nonholonomic constraints attached to the motion equations as initial conditions.

The attractors $(\theta,\dot{\theta},\omega_1)$, $(\Omega,\dot{\Omega},\omega_1)$, $(\theta,\dot{\theta},\Omega)$ and $(\omega_1,\dot{\omega}_1,\theta)$ are plotted in Figs. 6-9, respectively.

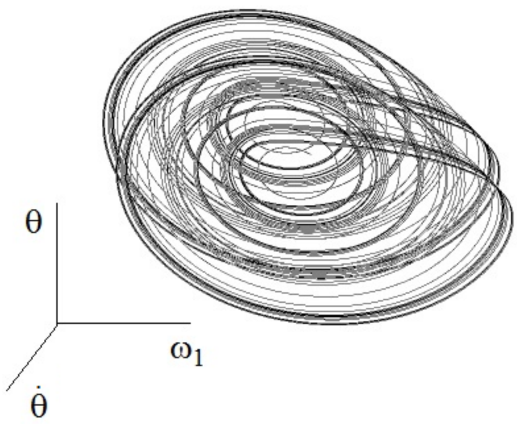

Fig. 6. The attractors $(\theta,\dot{\theta},\omega_1)$.

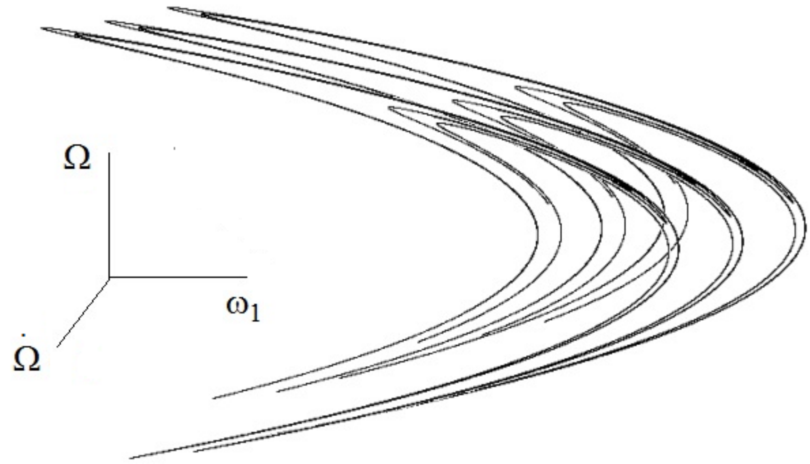

Fig. 7. The attractor $(\Omega,\dot{\Omega},\omega_1)$.

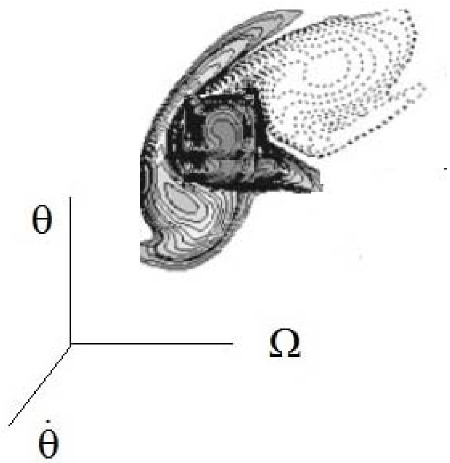

Fig. 8. The attractor $(\theta,\dot{\theta},\Omega)$.

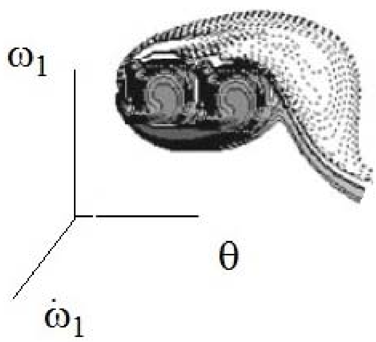

Fig. 9. The attractor $(\omega_1, \dot{\omega}_1, \theta)$.

4. CONCLUSIONS

In this paper, the rolling disk without sliding on a horizontal plane is investigated. The Chetaev nonholonomic constraints attached to the disk are treated as initial conditions. The Appell equations of motion of the disk are studied with respect to these initial conditions in order to capture the tendency to chaos of the disk. The solutions are found by using the cnoidal method.

When the angular velocity about the vertical is $\Omega > \sqrt{\dfrac{4g}{a}}$ and $\omega_1 = \dfrac{b\Omega}{a} \neq 0$, the rotating disk is rising, and the plane of the disc may rise first towards the vertical, and then falling towards the horizontal. Trajectories of O and A, respectively, are plotted nearly the disk falls flat, and also, the attractors are investigated.

The results of this paper have much significance in perfecting the cnoidal method to investigate the chaos behavior of dynamical systems.

REFERENCES

1. KRUPKOVA, O., *Geometric mechanics on nonholonomic submanifolds,* Communications in Mathematics, 18, 51–77, 2010.
2. JANOVA, J., *A Geometric theory of mechanical systems with nonholonomic constraints* , Thesis, Faculty of Science, Masaryk University, Brno, 2002 (in Czech).
3. JANOVA, J., MUSILOVA, J., *Non-holonomic mechanics : A geometrical treatment of general coupled rolling motion*, Int. J. Non-Linear Mechanics, 44, 98–105, 2009.
4. KOON, W.S., MARSDEN, J.E., *The Hamiltonian and Lagrangian approaches to the dynamics of nonholonomic system*, Reports on Mat. Phys., 40, 21–62, 1997.
5. KRUPKOVA, O., *Mechanical systems with nonholonomic constraints*, J. Math. Phys., 38, 5098–5126, 1997.
6. BLOCH, A.M., *Nonholonomic Mechanics and Control*, Springer Verlag, New York, 2003.
7. BRDICKA, M., HLADIK, *A.Theoretical Mechanics*, Academia, Praha, 1987 (in Czech).
8. BULLO, F., LEWIS, A.D., *Geometric Control of Mechanical Systems*, Springer Verlag, New York, Heidelberg, Berlin, 2004.
9. TEODORESCU, P.P., NICOROVICI, N.A.P., *Applications of the Theory of Groups in Mechanics and Physics* (Fundamental Theories of Physics) Springer, 2004.
10. TEODORESCU, P.P., *Mechanical Systems, Classical Models*, Volume 3: Analytical Mechanics (Mathematical and Analytical Techniques with Applications to Engineering) Springer, 2012.

11. VOINEA, R., VOICULESCU, D., CEAUŞU, V., *Mecanica*, Editura Didactica şi Pedagogica, Bucharest, 1983.
12. M. de LEON, MARRERO, J.C., D.M. de DIEGO, *Non-holonomic Lagrangian systems in jet manifold,* J. Phys. A: Math. Gen., 30, 1167-1190, 1997.
13. M. de LEON, MARRERO, J.C., D.M. de DIEGO, *Mechanical systems with nonlinear constraints*, Int. Journ. Theor. Phys., 36(4), 979-995, 1997.
14. GIACHETTA, G., *Jet methods in nonholonomic mechanics*, J. Math. Phys., 33, 16520-1655, 1992.
15. SWACZYNA, M., *Several examples of nonholonomic mechanical systems*, Communications in Mathematics, 19, 27-56, 2011.
16. NEIMARK, JU, I., FUFAEV, N.A., *Dynamics of Nonholonomic Systems*, Translations of Mathematical Monographs 33, American Mathematical Society, Rhode Island, 1972.
17. BATES, L., GRAUMANN, H., MACDONNELL, C., *Examples of gauge conservation laws in nonholonomic systems*, Rep. Math. Phys., 37 (3), 295-308, 1996.
18. BATES, L., SNIATYCKI, J., *Nonholonomic reduction*, Rep. Math. Phys., 32 (1), 99-115, 1992.
19. CORTES, J.M, MARTINEZ, S., OSTROWSKI, J.P., MCISAAC, K.A., *Optimal gaits for dynamic robotic locomotion*, Int. J. Robotics Research, 20 (9), 707-728, 2001.
20. KELLY, S.D., MURRAY, R.M., *Geometric phases and robotic locomotion*, J. Robotic Systems, 12 (6), 417-431, 1995.
21. OSTROWSKI, J.P., *Geometric Perspectives on the Mechanics and Control of Undulatory Locomotion*. PhD. Thesis, California Institute of Technology, 1995.
22. OSTROWSKI, J.P., BURDICK, J.W., *The geometric mechanics of undulatory robotic locomotion, Int. J. Robotics Research,* 17, 683-702, 1998.
23. OSTROWSKI, J.P., BURDICK, J.W., *Controllability tests for mechanical systems with symmetries and constraints*, J. Appl. Math. Comp. Sci., 7, 101-127, 1997.
24. CORTES, J.M., MARTINEZ, S., OSTROWSKI, J.P., ZHANG, H., S*imple mechanical control systems with constraints and symmetry*, SIAM J. Control Optim., 2002.
25. STĂNESCU, N.D., MUNTEANU, L., CHIROIU, V., PANDREA, N., Dynamical Systems. Theory and Applications, vol.2, Publishing House of the Romanian Academy, 2010.
26. STĂNESCU, N.D., MUNTEANU, L., CHIROIU, V., PANDREA, N., *Dynamical Systems. Theory and Applications*, vol.1, Publishing House of the Romanian Academy, 2007.
27. CENDRA, H., MARSDEN, J.E., RATIU, T.S., *Geometric mechanics, Lagrangian reduction and nonholonomic systems*, In Mathematics Unlimited-2001 and Beyond, eds. B. Enguist, W. Schmid, Springer- Verlag, New York, 2001, pp. 221-273.
28. WENDLANDT, J.M., MARSDEN, J.E., *Mechanical integrators derived from a discrete variational principle*, Physica D, 223-246, 1997
29. HAN, Y., WANG, X., ZHANG, M., JIA, L., *Special Lie symmetry and Hojman conserved quantity of Appell equations for a Chetaev nonholonomic system*, Nonlinear Dyn., 73, 357–361, 2013.
30. APPELL, P., *Sur une forme générale des équations de la dynamique*, 21**,** 310, 1900.
31. APPELL, P., *Traité de Mécanique Rationnelle*, vol. II, p. 335, Gauthier-Villars, Paris, 1953.
32. GIBBS, J.W., *On the Fundamental Formulae of Dynamics*, American Journal of Mathematics, 2, 49–64, 1879.
33. CORTES, J.M., *Geometric, Control and Numerical Aspects of Nonholonomic Systems*, 793 Lecture Notes in Mathematics, Springer, 2002.
34. CUSHMAN, R., DUISTERMAAT, SNIATYCKI, J., *Geometry of Nonholonomically Constrrained Systems*, Advanced Series in Nonlinear Dynamics, vol.20, World Scientific, 2009.
35. MUNTEANU, L, DONESCU, ŞT., *Introduction to Soliton Theory*: *Applications to Mechanics,* Book Series Fundamental Theories of Physics, vol.143, Kluwer Academic Publishers, Dordrecht, Boston (Springer Netherlands) 2004.
36. JIA, L.Q., SUN, X.T., ZHANG, M.L., WANG, X.X., XIE, Y.L., *A type of new conserved quantity of Mei symmetry for Nielsen equations*, Acta Phys. Sin., 60, 084501, 2011.
37. CAI, J.L., SHI, S.S., FANG, H.J., XU, J., *Conformal invariance for the nonholonomic constrained mechanical system of non-Chetaev's type*, Meccanica, 47, 63–69, 2012.
38. JIA, L.Q.,WANG, X.X., ZHANG, M.L., HAN, Y.L., *Special Mei symmetry and approximate conserved quantity of Appell equations for a weakly nonholonomic system*, Nonlinear Dyn., 69, 1807–1812, 2012.

39. CAI, J.L., *Conformal invariance and conserved quantities of general holonomic systems*, Chin. Phys. Lett., 25, 1523-1526, 2008.
40. FU, J.L., NIE, N.M., HUANG, J.F., *Noether conserved quantities and Lie point symmetries of difference Lagrange-Maxwell equations and lattice,* Chin. Phys. B, 18, 2634–2641, 2009.
41. FANG, J.H., *A kind of conserved quantity of Mei symmetry for Lagrange system*, Acta Phys. Sin. **58**, 3617–3619, 2009.
42. CAI, J.L., *Conformal invariance and conserved quantities of Mei symmetry for general holonomic systems*, Acta Phys. Sin., 58, 22–27, 2009.
43. CAI, J.L., *Conformal invariance and conserved quantities of Mei symmetry for Lagrange systems*, Acta Phys. Pol. A, 115, 854–856, 2009.

Received March 12, 2018