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THE BEHAVIOUR OF THE SELF-FOCUSING AND SELF-DEFOCUSING MATERIALS - A REVIEW

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Abstract The paper is a short review on a special class of Cantor-like materials under resonance and pulse-mode conditions. These materials are highly nonlinear and hysteretic, exhibiting the intriguing phenomena like self-focusing and self-defocusing. The balance between the dispersion and concentration determines the behavior of these materials.

Key words: Self-focusing, Self-defocusing, Cantor like structure, Solitons.

1. INTRODUCTION

The paper is a review of the self-focusing and self-defocusing materials. Dispersion, concentration, attenuation and amplification of waves are special properties of materials which depend on the internal structure of the material. The balance between these phenomena in the global behavior of the material determine the self-focusing and self-defocusing features [1, 2].

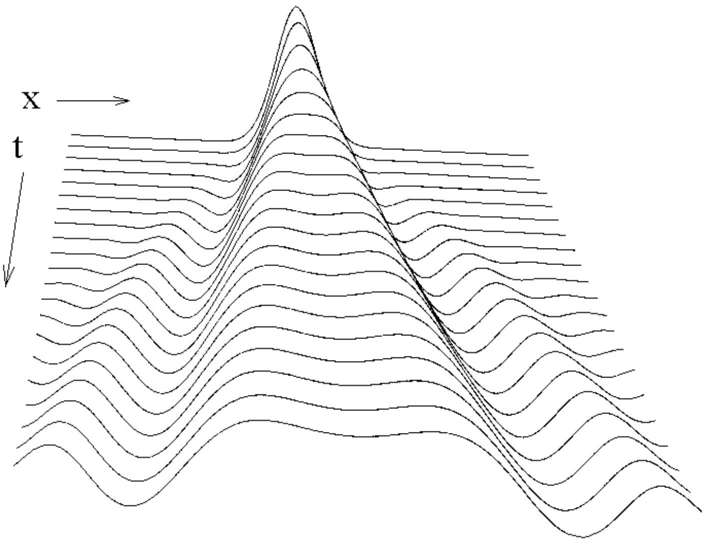
Dispersion of a Gaussian pulse and its attenuation are displayed in Fig. 1 and Fig. 2, respectively [3]. In the nature each effect has its opposite. Thus, the opposite of the dispersion is the energy concentration. To illustrate this feature, we revers the time axis of Fig.1. and the detected image represent the concentration. Mechanical radiation is the result of the dispersive transmission of the energy under the form of waves through the material of different frequencies and different velocities. This includes acoustic radiation such as seismic waves.

The concentration of the energy takes place in certain parts of the material and is represented by focusing of the energy as if it had gone through internal lenses.

The opposite of attenuation is amplification. Changing the direction of the time axis in Fig. 2 we get amplification [3]. Amplification is achieved in active systems where there is an input of energy, i.e. a pumping of energy from subsystems or sub-processes. Amplification also existed in disturbed systems or due to volume forces.

Nature is far from being homogeneous, conservative, etc. In reality, the energy is often not balanced in dynamic systems, it is not preserved, so there is either outflow of energy (energy is lost) or energy influx. The first case, outflow of energy has been studied for a long time and the mechanism of attenuation (material attenuation or geometric attenuation) has been well analyzed and modeled. However, the energy influx is not so well understood and modeled.

Engelbrecht published in 1991 an introduction to asymmetric solitary waves [4] in which he analyzed some energy-intensive systems based on the concept of dilaton introduced in 1983 by Petrov [5] and Zhurkov [6] in modeling cracked media. The dilaton was used by Engelbrecht to explain the possible amplification of seismic waves.



. Fig. 1. Dispersion of a Gaussian pulse.

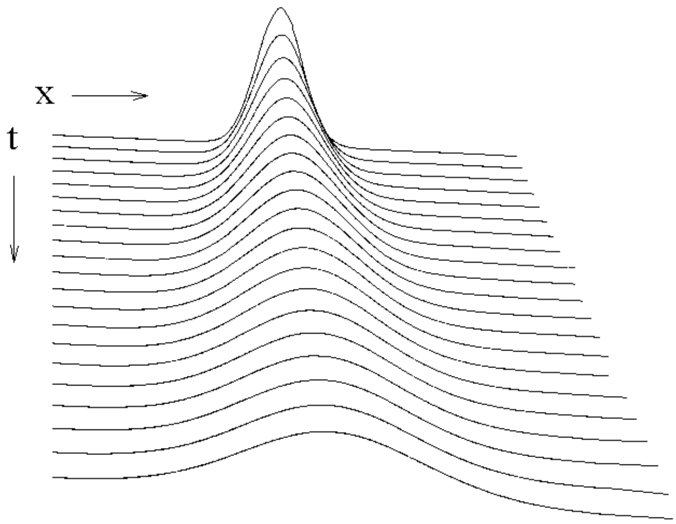


Fig. 2. Atenuation of a Gaussian pulse.

The basic idea is that the weakening or loss of structural bondings (atomic bonds or joints between structural blocks, layers, phases, etc.) leads to fluctuations of internal energy. Such a short-term fluctuation of internal energy density is called dilaton. The dilaton absorbs the energy from the surrounding environment. Thermodynamic considerations do not allow an infinite increase of the energy absorbed into a dilaton. Therefore, there is a threshold, a maximum for this energy. When this value is reached, the dilaton is self-destructed, cracked by releasing the accumulated energy.

The dilaton absorbs or radiate energy. This process is controlled by the intensity of the waves propagating through the medium. Low intensity waves yield some of their energy to the dilatations and as a result the attenuation occurs. Waves of high intensity cause the breakdown of the dilaton and its energy is transferred to the waves, gaining the amplification. The dynamic behavior of the materials results from how these effects and their counterparts are balanced or not during the process. Thus, the solitary wave is a manifestation of the balance between the non-linear effect and the dispersive effect.

In the case of an earthquake, there is a connection between large blocks and not between atoms. These may be macro-dilatons. When the maximum value of the energy that characterizes the dilaton is small, the process is attenuated for any wavelength, and this is the case of dissipative materials. When this value is higher, amplifications appear, this value depending on the material.

In 1989 The Euromech International Symposium entitled "Nonlinear Underground in Active Media" has analyzed the main causes of the development of dilatons. They occur due to the energy pumping from sub-systems and sub-processes through wave propagation. They may also appear in disturbed systems. A classic example of dilatation is the combustion of luminescence. Burning is described by a non-linear subset and the firing rate depends on the calorific energy given. Another example of the dilaton is the nerve-electromagnetic pulse that accumulates energy from the ions.

In its path, a wave may encounter dilated areas, or areas in which dilatations are absent, or randomly distributed dilatons. Sadovski and Nikolaev [9] showed experimentally in 1982 that in terrestrial crust there are regions with high activity characterized by phase transitions, geological faults, defective interfaces, tectonic stress concentrations, in which the propagation of seismic waves is amplified. They uniaxially compressed in situ a linear inhomogeneity.

The linear non-homogeneity required by the tectonic voltage led to high voltage concentrations, i.e., to the formation of a dilaton that absorbed this energy. When the dilaton brokes all its energy was released to the field from where waves were gained.

The dilatons described by Zhurkov and Petrov in 1983 have been applied in the theory of material breakage, where atomic laws are important. We can call these micro-dilaton fluctuations.

Evidence of concentration and amplification has been observed in seismic events, when unexpected deteriorations and several damages occur in some places in the world with magnitudes higher than in the epicenter [7], or in the breaking of materials, the so-called crazy fracture [8, 9], which does not resemble any known pattern of breaking the materials.

The city of Santa Monica (21 kilometers from the epicenter of the Northridge earthquake, magnitude 6.7) experienced in 1994 an abnormally concentrated seismic intensity and a severe deterioration with the intensity of Mercalli 9, an intensity as high as that which occurred in the vicinity of the epicenter [7]. Seismic recordings in the replies suggest that the damage resulting from the seismic wave focusing of several underground acoustic lenses at depths of about 3 kilometers, formed by defects that bound the northwestern edge of the Los Angeles basin. The amplification was the highest for high frequency waves and was less powerful at lower frequencies, which corresponds to the focus theory and finite-difference simulations.

In this work, we review on a special class of materials based on the Cantor-like structure which exhibits the self-focusing and self-defocusing phenomena. These phenomena can be explained by occurrence of two kind of vibration regimes in such materials: a localised-mode (fracton) regime of vibrations characterized by localised vibrations on certain small parts of the material, and an extended-vibration (phonon) regime essentially extended to the whole material. Self-focusing and self-defocusing are explained by nonlinear coupling between the extended-vibration (phonon) and the localized-mode (fracton) regimes in such materials.

The classical continuum theory is inadequate for the treatment of such phenomena. The effect of internal structure becomes important in materials subjected to loadings of small wavelength and high frequency. When the wavelength is comparable with the average micro-grain size, the motion of the grains must be taken into account because the grains exhibit extra independent internal degrees of freedom for local rotations. The micro-grains are allowed to rotate independently without stretch. The behavior of waves in such media exhibits new features as dispersion, concentration, focusing, self-focusing and self-defocusing, respectively. The couple stresses theory and the soliton theory are applied to understand these features [8-13]. Previous works discussing the focusing and self-focusing phenomena are reported in [14-18].

We summarize the introduction with the motion equations of waves in a dilatonic layer localized into a medium [4]

$$(\sigma_{KL}x_{k,L})_{,K} + \rho_0(f_k - A_k) = 0\,, \tag{1}$$

where ρ_0 is the density of the undeformed medium, σ_{KL} is the Kirchhoff stress tensor, f_k body forces, A_k accelerations, and x_k the Euler coordinates. Coma means differentiation of the specified variable with respect to the Lagrange coordinates X_K.

Constitutive law of the material is

$$\sigma_{KL} = \sigma_{KL}(E_{KL}, \frac{\partial E_{KL}}{\partial t})\,, \tag{2}$$

where E_{KL} is the Green strain tensor, The dilatonic constitutive law represents the dependence of the body forces on E_{KL}

$$f_k = f_k(E_{KL})\,. \tag{3}$$

From (1) we obtain the motion law express in terms on the transversal displacement U_2

$$\frac{\partial^2 U_2}{\partial t^2} = c_{s2}^2 \frac{\partial^2 U_2}{\partial X^2} + c_{s2}^2 m(\frac{\partial U_2}{\partial X_1})^2 \frac{\partial^2 U_2}{\partial X_1^2} + c_{s2}^2 l_0^2 \frac{\partial^4 U_2}{\partial X_1^4} + f_2\,, \tag{4}$$

where X_1 is the longitudinal direction of the layer, and m a parameter which depend on the elastic constants of the layer, c_{si}, $i = 1,2$, the velocities of the transversal waves in the layer ($i = 1$) and the mediu ($i = 2$)

$$\omega^2 = c_{s2}^2(k^2 - l_0^2 k^4)\,,\; l_0^2 = h^2(\frac{c_{s2}^2}{c_{s1}^2} - 1)^2 \frac{\nu_1^2}{\nu_2^2} > 0\,, \tag{5}$$

where) ω is the angular frequency, k is the wave number, and ν_i, $i = 1,2$, the Poisson ratios in the layer and medium, respectively.

Eq.(4) can be written as

$$\frac{\partial u}{\partial \tau} - \frac{m}{2\varepsilon^2 c_{s2}^2} u^2 \frac{\partial u}{\partial \xi} + \frac{1}{2}\frac{l_0}{\varepsilon}\frac{\partial^3 u}{\partial \xi^3} + \frac{1}{2\varepsilon\, c_{s2}} f_2 = 0\,, \tag{6}$$

with $\xi = c_{s2}t - X_1$, $\tau = \varepsilon^2 X_1$, $u = \frac{\partial U_2}{\partial t}$. The small parameter ε depends on $\frac{\partial U_2}{\partial X_1}$.

The dilatonic constitutive law (3) is written as

$$f_2 = -(b_1 U_{2,1} - b_2 U_{2.1}^2 + b_3 U_{2,1}^3)\,, \tag{7}$$

with b_i, $i = 1,2,3$, positive constants, or

$$f_2 = B_1 u - B_2 u^2 + B_3 u^3\,, \tag{8}$$

and $B_1 = \dfrac{b_1}{c_{s2}}$, $B_2 = \dfrac{b_2}{c_{s2}^2}$, $B_3 = \dfrac{b_3}{c_{s2}^3}$.

By introducing the dimensionless variables

$$\tau = \frac{u}{u_0}, \quad \sigma = \frac{a_1 u_0^2 \tau}{\tau_0}, \quad \zeta = \frac{\xi}{\tau_0}, \quad a_1 = \frac{1}{2}|m|\varepsilon^{-2} c_{s2}^{-2}.$$

with u_0 maximum amplitude of the velocity, and τ_0 wavelength, Eq. (6) becomes

$$\frac{\partial v}{\partial \sigma} - \mathrm{sign}(m) v^2 \frac{\partial v}{\partial \zeta} + \mu \frac{\partial^3 v}{\partial \zeta^3} + f(v) = 0\,, \tag{9}$$

where

$$\mu = \frac{\varepsilon l_0^2 c_{s2}^2}{\tau_0 u_0^2 |m|}, \quad f(v) = \beta_1 v - \beta_2 v^2 + \beta_3 v^3\,, \tag{10}$$

$$\beta_1 = Q\frac{B_1}{u_0^2} > 0\,, \ \beta_2 = Q\frac{B_2}{u_0} > 0\,, \quad \beta_3 = QB_3 > 0\,, \quad Q = \frac{\tau_0 \varepsilon c_{s2}}{|m|}. \tag{11}$$

Eq. (9) describes the motion of a long transversal wave in the layer, generated by the body force f. Cubic nonlinearity of this force reduces Eq. (9) to a modified KdV equations, with soliton solutions [10].

The soliton has the form of a localized pulses that conserve its properties even after interaction with other waves, and then acts somewhat like a particle. The KdV equation has an infinite number of local conserved quantities, an infinite number of exact solutions expressed in terms of the Jacobi elliptic functions (cnoidal solutions) or the hyperbolic functions (solitons), and the simple formulae for nonlinear superposition of explicit solutions. Such equations are integrable or exactly solvable [10].

Fig. 3 represents this solution of the wave with the amplitude A_0.

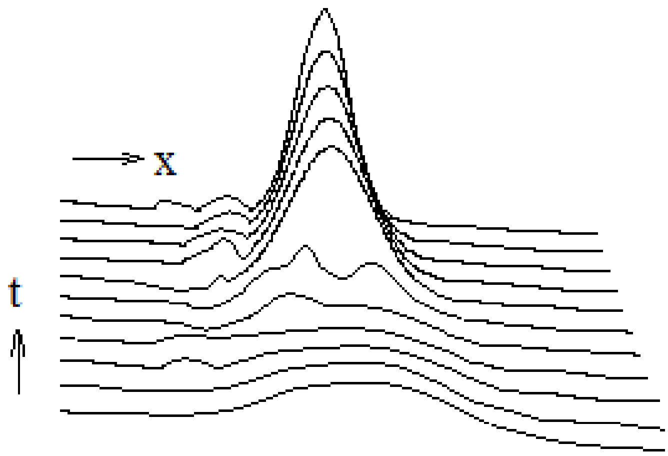


Fig. 3. Soliton solution of the problem.

The analysis shows that for greater values of the parameter $\tau = \tau_c = \left(\frac{4}{45}\frac{a_2}{\mu}\right)^{-1/2}$, $a_2 = \frac{8}{5}b_2 > 0$ there exists a threshold value for the wave amplitude $A_{cr} = \frac{a_2}{15\mu\,|\,a_0\,|}$, so that for $A_0 < A_{cr}$ the amplitude of the soliton is decreasing. For $A_0 > A_{cr}$, the amplitude of the soliton is explosively increasing. The nonlinear and dispersive effects are in equilibrium or not, depending on the force f and the initial conditions. In equilibrium, the wave is moving through the layer, unchanged, with the same speed and the same shape, keeping its identity.

In special conditions, the force f destroys this equilibrium and the soliton receives energy from the destroyed dilator. Fig. 4 illustrates this phenomenon for ratio $r_0 = \frac{A_0}{A_{cr}} = 0.98$. Figure plots the formation of the soliton with high amplitude, and then the attenuation in time of the soliton amplitude.

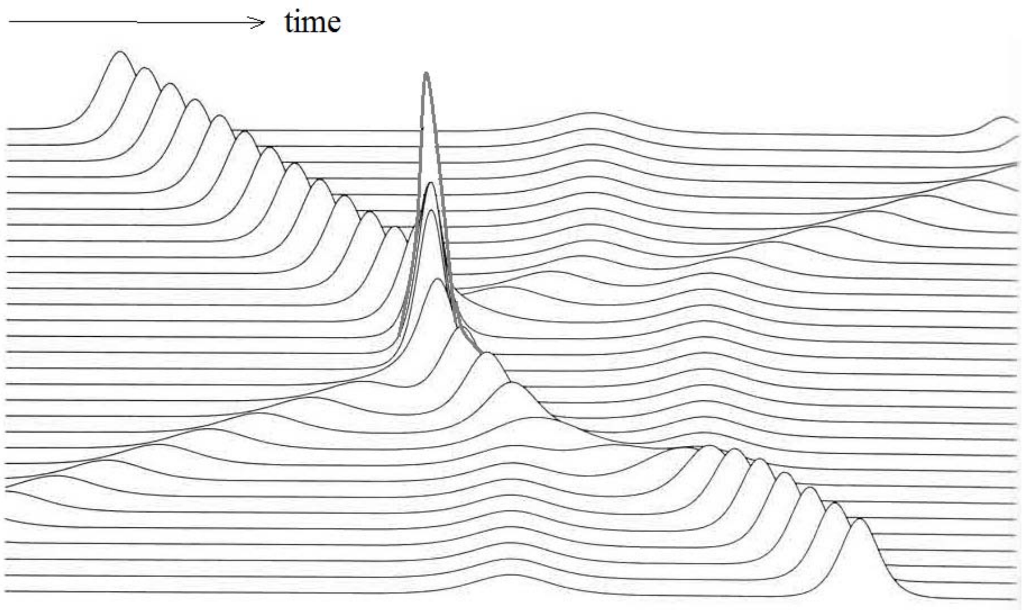


Fig. 4. The amplification of the soliton.

2. MATERIALS WITH CANTOR-LIKE STRUCTURE

Craciun et al. [19] and Allipi et al. [20-22] experimentally put into evidence special properties of 1D artificial piezoelectric plates with Cantor-like structure, as compared to the corresponding homogeneous and periodical plates. They highlighted a low threshold for subharmonic generation of ultrasonic waves, showing that the interaction is done by certain favorable frequency and spatial matching of vibrational modes.

We consider a plate consisted by alternating elements of piezoelectric ceramics (PZ) and epoxy resin (ER), following a triadic Cantor sequence (31 elements) [23, 24]. The length of the plate is l, the width of the smallest layer is $l/81$ and the thickness of the plate is h (Fig. 5).

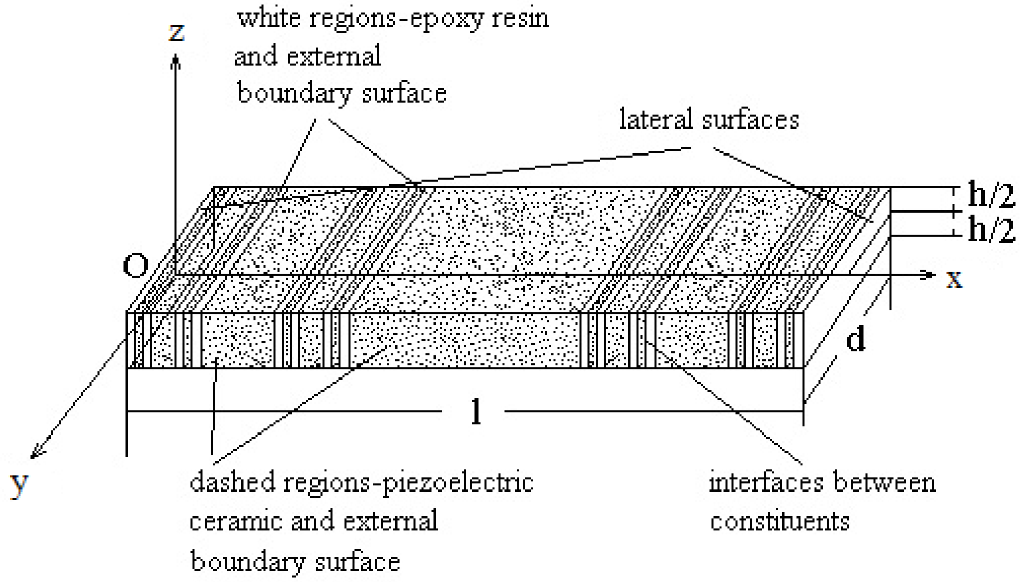


Fig. 5. The plate with Cantor like structure.

Table 1 shows the material constants for piezoceraamics and epoxy resin, respectively. Table 2 shows the computed frequencies and the errors obtained by the eigenvalue-problem.

Table 1 The material constants for piezoelectric ceramics and epoxy resin

	piezoelectric ceramics	epoxy resin
λ	71.6 GPa	42.31 GPa
μ	35.8 GPa	3.76 GPa
A	-2000 GPa	2.8 GPa
B	-1134 GPa	9.7 GPa
C	-900 GPa	-5.7 GPa
$\overline{\varepsilon}$	4.065 nF/m	-
$\overline{\varepsilon}_1$	2.079 nF/m	-
e_1	-0.218 nm/V	-
$\overline{e}_1 = \overline{\overline{e}}_1$	-0.435 nm/V	-

ρ	7650 Kg/m^3	1170Kg/m^3

Table 2. Estimation results: computed eigenfrequencies.

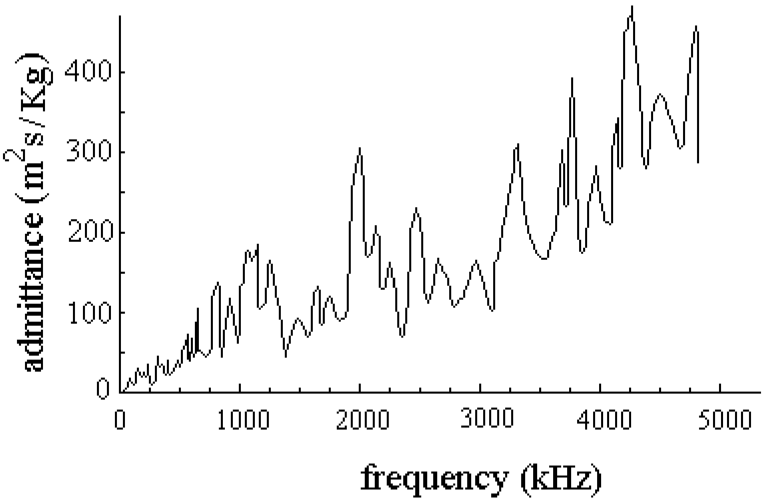
$\omega_n/2\pi$	100.2 ± 0.05	167 ± 0.01	217.1 ± 0.03	250.5 ± 0.1	334 ± 0.01	367.4 ± 0.01	417.5 ± 0.1	501 ± 0.02	584.5 ± 0.03
	617.9 ± 0.01	668 ± 0.03	835 ± 0.06	935.2 ± 0.06	1085.5 ± 0.1	1169 ± 0.07	1269.2 ± 0.02	1503 ± 0.05	1670 ± 0.4
	1770.2 ± 0.2	1987.3 ± 0.12	2120.9 ± 0.02	2250 ± 0.1	2471.6 ± 0.3	2655.3 ± 0.01	2672 ± 0.01	2972.6 ± 0.2	3340 ± 0.4
	3540.4 ± 0.04	3577.4 ± 0.02	3690.7 ± 0.01	3774.2 ± 0.15	3974.6 ± 0.07	3991.3 ± 0.24	4241.8 ± 0.07	4250 ± 0.03	4291.9 ± 0.06
	4322 ± 0.04	4525.7 ± 0.2	4655 ± 0.1	4698.6 ± 0.02	4766 ± 0.2	4798.4 ± 0.03	4826.3 ± 0.01	4856 ± 0.04	4881.7 ± 0.04
	4899.4 ± 0.01	4901 ± 0.04	4943.2 ± 0.1	5003.5 ± 0.1	5019.4 ± 0.15	5122.3 ± 0.07	5146.6 ± 0.16	5233 ± 0.1	5256.9 ± 0.3
	5298.6 ± 0.1	5308 ± 0.06	5310.6 ± 0.02	5319.5 ± 0.5	5344 ± 0.15	5367.7 ± 0.51	5401.9 ± 0.55	5423 ± 0.01	5436.7 ± 0.01

Resonant vibration modes are excited by applying an external electric field $\overline{E}_1 = \overline{E}_3 = \overline{E}^0 \exp(i\omega_0 t)$ on both sides of the plate with $\omega = \omega_n$. The length of the plate is 81cm and the thickness 5cm.

In Fig. 6 the admittance curve ($k/\rho\omega$ vs. $\omega/2\pi$) in the linear regime ($\overline{E}^0 \cong 0.1V$) with peaks for frequencies $\omega = \omega_n$. We see a good agreement between the eigenfrequencies given by this curve and the theoretical results. If $\overline{E}^0$ is increases above a threshold value $\overline{E}^0_{th}$ = 5.27 V the $\omega/2$ subharmonic generation is observed.

Two kind of vibration regimes are found: a localised-mode (fracton) regime represented in Fig. 7 for $\omega/2\pi$ = 1169 kHz, 2672 kHz and 3340 kHz and an extended-vibration (phonon) regime represented in Fig. 8 for $\omega/2\pi = 4175$ kHz and 4250 kHz. The fracton vibrations are mostly localised on a few elements, while the phonon vibrations essentially extend to the whole plate. In the case of a periodical plate the dispersion prevents good frequency matching between the fundamental and appropriate subharmonic modes.

For the homogeneous plate the mismatch $\omega_n - \omega/2$ is due to the symmetry of fundamental modes with respect to x. Only symmetric odd n can induce a subharmonic, but never $\omega/2$



coincides with a plate vibration mode. For a Cantor plate, we have obtained the same result as Craciun and Alippi [19-22]: given a normal mode ω_n, for excitation at $\omega = \omega_n$ the value of the expected threshold E_{th} i. e. the ability of generating the $\omega/2$ subharmonic, is determined by the existence of a normal mode with: (i) small frequency mismatch $\omega_n - \omega/2$, and, (ii) large spatial overlap between the fundamental and subharmonic displacement field.

Fig. 6 The admittance-frequency curve for the Cantor plate.

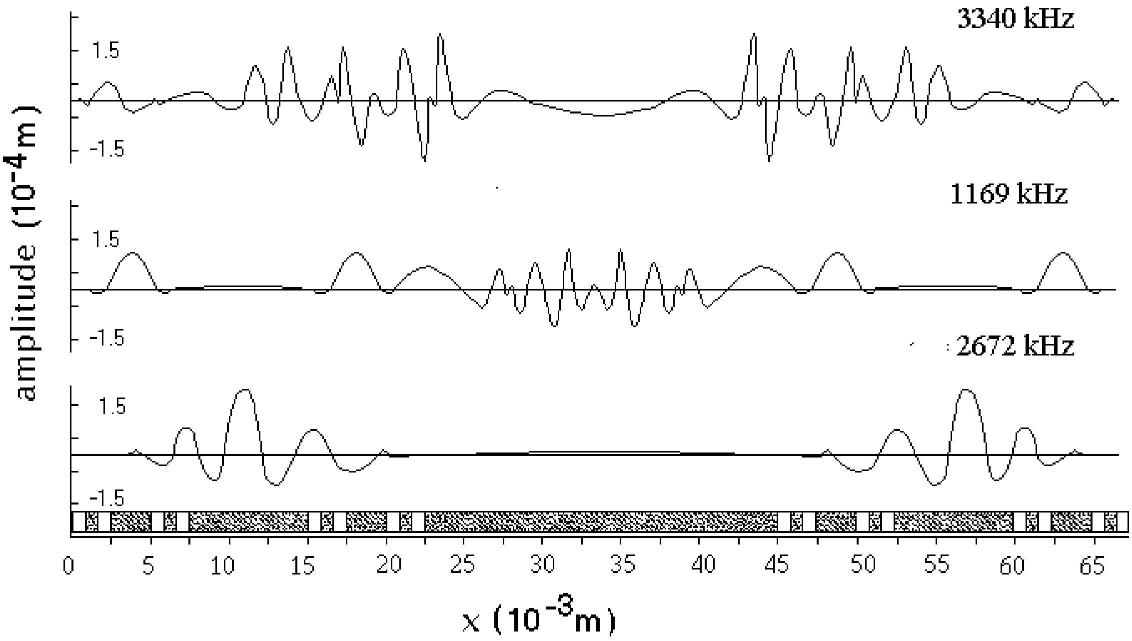


Fig. 7. The fracton localised vibrations for $\omega/2\pi$ =1169 kHz, 2672 kHz and 3340 kHz.

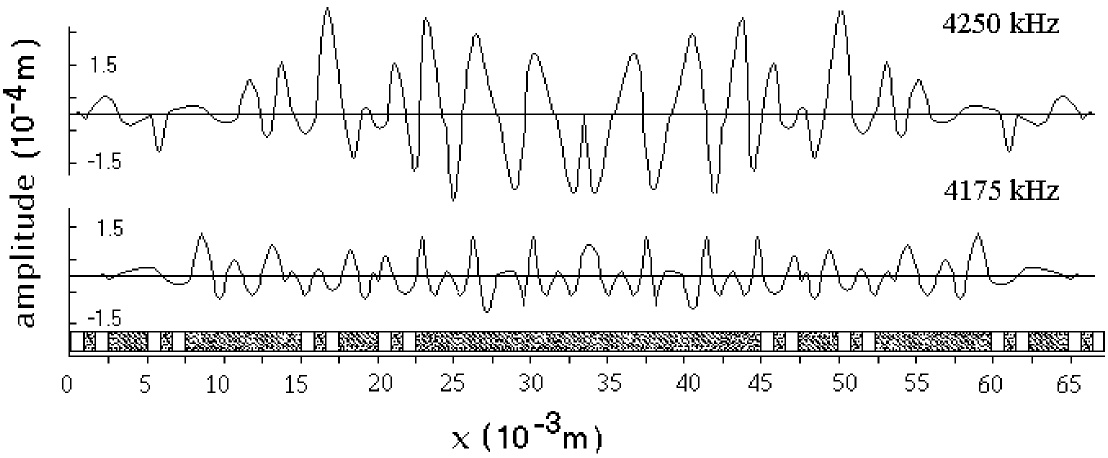


Fig. 8. The phonon extended vibrations for $\omega / 2\pi = 4175$ kHz and 4250 kHz.

Our results show that the fraction-phonon vibrations interaction gives rise to high amplitude solitons in the middle of the plate. This represents the self-focusing phenomenon. Fig. 9 shows a soliton of small amplitude at time = 40 sec, for 2672 kHz. After 10 sec a soliton of high amplitude occurs in the middle of the plate. Fig. 10 shows two solitons of high amplitude in the same location at time = 60 sec. After 5 sec the amplitudes decrease. Decreasing of amplitude represents the self-focusing phenomenon. Fig. 11 shows two solitons of low amplitude at the middle of the plate at time = 70 sec. After 30 sec. the wave in the middle of the plate return to that of time = 40 sec.

Details on the soliton theory which stay to the base of the explanation of the self-focusing and self-defocusing phenomena in materials are found in [25-38].

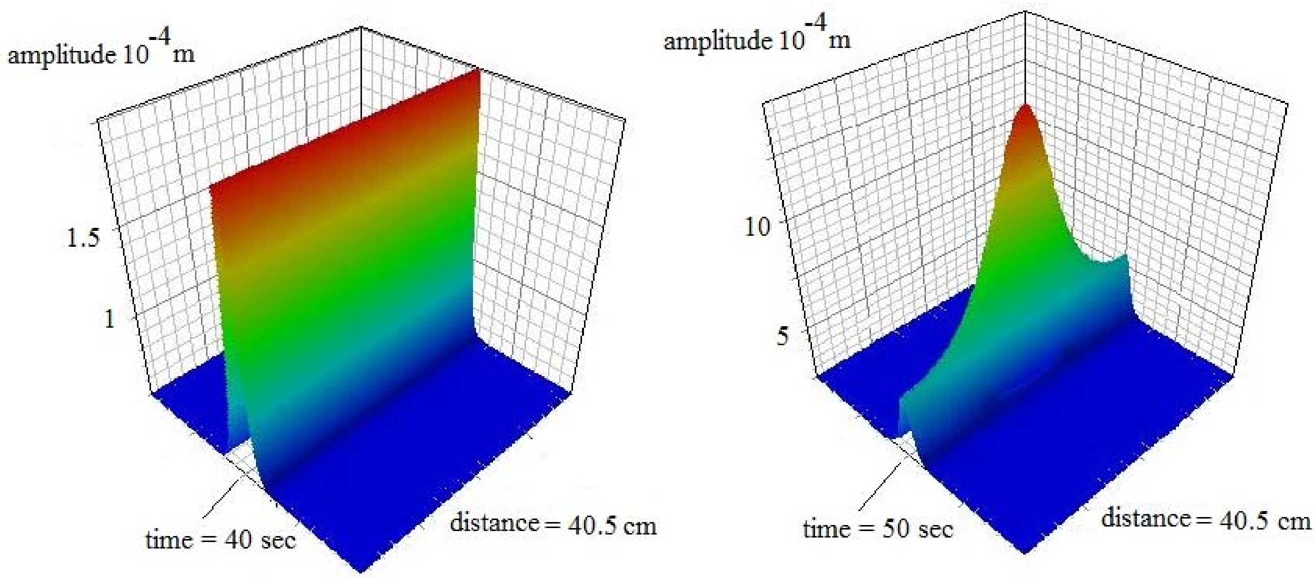


Fig. 9. Soliton of small amplitude at time = 40 sec. After 10 sec a soliton of high amplitude occurs in the middle of the plate.

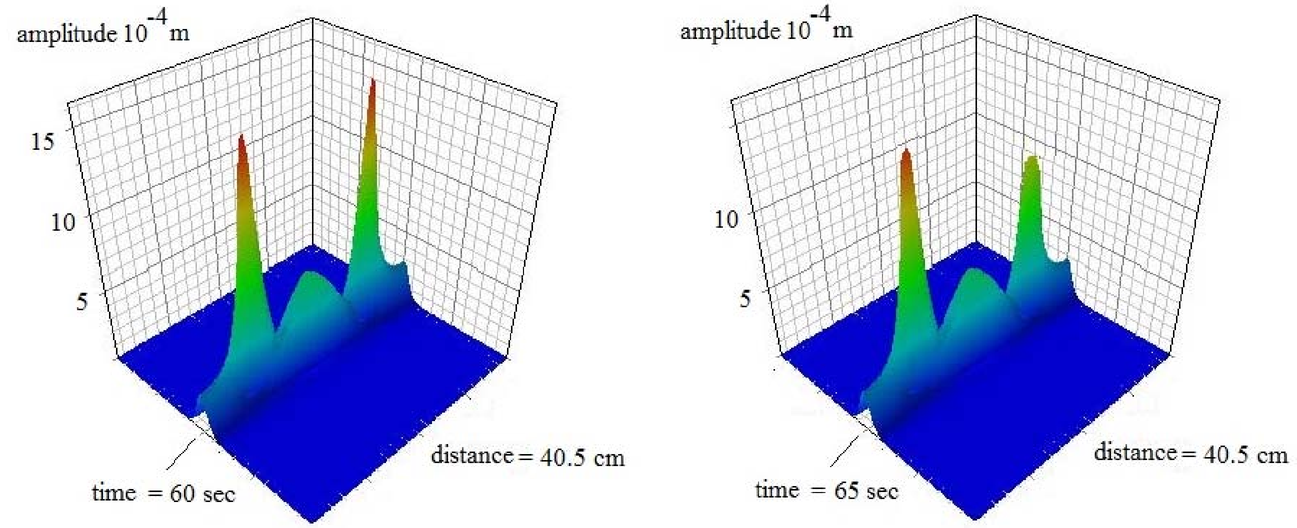


Fig. 10. Self-focusing by occurring of two solitons of high amplitude in the middle of the plate at time = 60 sec. After 5 sec the amplitudes decrease.

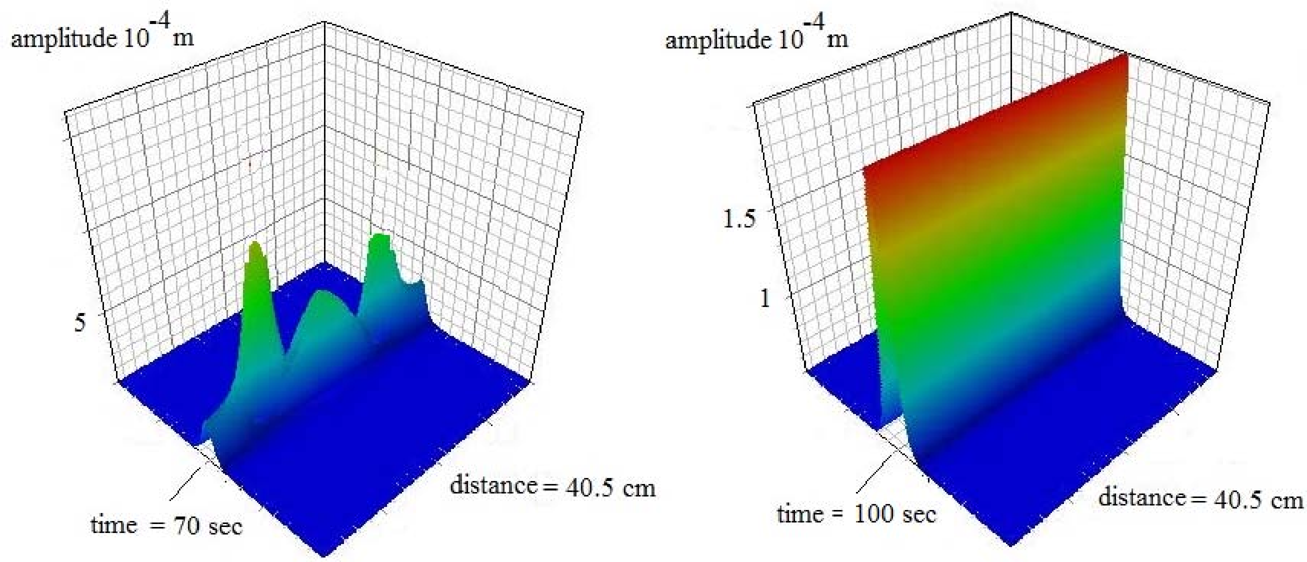


Fig. 11. Two solitons of low amplitude at the middle of the plate at time =70 sec. After 30 sec, the wave is self-defocusing.

3. CONCLUSIONS

In this paper, a short review on a special class of Cantor-like materials under resonance and pulse-mode conditions is presented. These materials are highly nonlinear and hysteretic, exhibiting the self-focusing and self-defocusing phenomena. The balance between the dispersion and concentration determines the behavior of these materials.

The phenomena of self-focusing and self-defocusing can be explained by generatione of two kind of vibration regimes in materials: a localised-mode (fracton) regime of vibrations characterized by localised vibrations on certain small parts of the material, and an extended-vibration (phonon) regime essentially extended to the whole material. The nonlinear coupling between the extended-vibration (phonon) and the localized-mode (fracton) regimes in such materials is the starting point in explanation of these phenomena.

The effect of internal structure becomes important in materials subjected to loadings of small wavelength and high frequency. When the wavelength is comparable with the average micro-grain size, the motion of the grains must be taken into account because the grains exhibit extra independent internal degrees of freedom for local rotations. The micro-grains are allowed to rotate independently without stretch.

The topic is far from being fully solved. The subject is open and offers study opportunities to both researchers and experimenters

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