Romanian Journal of Mechanics
Volume 3, Issue 2/2018, pp 48-63, ISSN 2537 - 5229, rjm.journals.srmta.ro

FINITE ELEMENT ANALYSIS OF FLEXIBLE MULTIBODY SYSTEMS. AN OVERVIEW

Sorin VLASE
Transylvania University of Brașov, Department of Mechanics

Corresponding author: Sorin VLASE, E-mail: svlase@unitbv.ro

Abstract The paper aims to present a summary of the results obtained in the field of multicorp systems with elastic elements. The first works in the field appeared in the 70's and until now numerous works have been published that contribute to the topic studied. The big problem of the obtained results is the difficulty of numerical approach to such a problem. If, from a theoretical point of view, it can be said that the results obtained so far are satisfactory, when the concrete numerical calculation is made, the results are more than disappointing. Finding effective ways to integrate and get the answer in time has remained a challenge for researchers.

Key words: multi-body system, finite element method, linear elastic elements, Lagrange's equations, three-dimensional motion, one-dimensional finite element

1. INTRODUCTION

Over the past fifty years, the high speeds of the various machine components and the great forces with which the various mechanisms operate make the elements' elasticity significantly influence their operation. Resonance phenomena and loss of stability are classical forms of manifestation of elastic properties. At the design stage it is necessary to anticipate and remove these unfavorable effects. As a result, a study of these phenomena is required. The first papers in the field of these systems were made on mathematical models using theories of elasticity theory. Unfortunately, the differential equations obtained and describing the evolution in time of the system are difficult to solve, even if standard numerical methods of solving are used. The finite element method has proven to be the most powerful way to solve this problem. The advantages of this method have been presented in a series of papers as [1], [3], [5]- [9], [11], [15], [24], [34]-[36].

The first paperss in the field studied the systems in which we have an elastic element with a plane motion and were then developed for a more comple mechanisms systems with plane motion [5], [9], [18], [19], [22], [27], [39], with the deformable elements. In the paper [10], [13], [20], the results are being synthetized. Particular effects are studied using more complex model [12], [23], [45], [46].

The research carried out in this field approached aspects regarding the calculation, experimental checks and control in case of simple mechanical systems ([8], [17], [24],). The influence of damping, the stability or the use of some composite materials [22], [27], [45], [46] or thermal problems (see [12]) has been studied. The main difficulty consisted in the symbolic representation of the equations of motions and in finding methods of integration. Such models for two- and three-dimensional motions have been developed by [34]. Experimental studies validated the chosen model [32]. The development of the field has also resulted in theoretical results in the field of mechanics of multicorp systems and the elimination of liaison forces in motion equations [4], [14], [16], [20], [21], [25], [26], [29]-[31], [33], [37], [38]. A systematic presentation of the results was done in [41]-[43].

If the obtained motion equations are compared to those obtained in case of the study of the steady state response one can find that additional terms occur. They are due to the relative motion of nodal coordinates relative to the mobile coordinate systems attached to the moving bodies - Coriolis effects – and to the change of stiffness determined by the accelerations field. The inertia terms will be also modified if one takes into account the inertia effects due to the motion of the finite element relative to the coordinate system attached to the moving bodies.

The determination of the equations of motion represents the first step in solving this kind of problems. The next step is to relate the equations of motion to the global coordinate system and to assemble them for obtaining the set of differential equations which will describe the evolution in time of the mechanical system with elastic elements in terms of independent nodal coordinates.

2. FINITE ELEMENT ANALYSIS OF A MULTIBODY SYSTEM WITH FLEXIBLE ELEMENTS

The change from the local coordinate system to the global coordinate system is accomplished through the rotation matrix. The relationships between the components of a vector expressed in the two coordinate systems are

$$x_{1i} = r_{ij} x_j \ , \quad i, j = 1, 2, 3. \tag{1}$$

The components of the rotation matrix r_{ij} define the components of the unit vectors of the local coordinate system $Oxyz$ refer to the global coordinate system $Ox_1y_1z_1$. The derivatives of these values allow us to get the angular velocities and angular accelerations. In our paper we shall express these derivatives according to [44]. The orthogonality conditions of the unit vectors lead to

$$r_{ij} r_{kj} = r_{jk} r_{ji} = \delta_{ij} \, , \tag{2}$$

where δ_{ij} is the Kronecker delta. If we differentiate this equation it will results

$$\dot{r}_{ij} r_{kj} + r_{ij} \dot{r}_{kj} = 0, \quad i, k = 1, 2, 3 \, , \tag{3}$$

with the notations

$$\Omega_{ik} = \dot{r}_{ij} r_{kj} \, , \tag{4}$$

rel. (3) becomes

$$\Omega_{ik} + \Omega_{ki} = 0 \, . \tag{5}$$

The skew-symmetric tensor Ω_{ik} is the so-called operator angular velocity (its components are expressed in the global coordinate system). To this operator corresponds the angular velocity vector defined by

$$\Omega_1 = \Omega_{32} = -\Omega_{23}, \quad \Omega_2 = \Omega_{13} = -\Omega_{31}, \quad \Omega_3 = \Omega_{21} = -\Omega_{12}. \tag{6}$$

We shall also have the angular acceleration skew symmetric operator, defined a:

$$\mathrm{E}_{ik} = \dot{\Omega}_k = \ddot{r}_{ij} r_{kj} + \dot{r}_{ij} \dot{r}_{kj} \ . \tag{7}$$

To this corresponds the angular acceleration vector defined by

$$\mathrm{E} = \mathrm{E}_{32} = -\mathrm{E}_{23}, \mathrm{E}_2 = \mathrm{E}_{13} = -\mathrm{E}_{31}, \mathrm{E}_3 = \mathrm{E}_{21} = -\mathrm{E}_{12} \ . \tag{8}$$

After some elementary calculations we shall have

$$\mathrm{E}_{ik} = \Omega_{ik} = \ddot{r}_{ij} r_{kj} + \dot{r}_{ij} \dot{r}_{kj} = \ddot{r}_{ij} r_{kj} + \dot{r}_{ij} r_{jl} r_{ml} \dot{r}_{km} = \ddot{r}_{ij} r_{kj} - \Omega_{il} \Omega_{lk} \, , \tag{9}$$

from where

$$\ddot{r}_{ij} r_{kj} = \mathrm{E}_{ik} + \Omega_{il} \Omega_{lk} \ . \tag{10}$$

It is possible to express the angular velocity and acceleration vectors in the local coordinate system using the relations

$$\omega_i = r_{ij} \Omega_j, \quad \varepsilon_i = r_{ij} \mathrm{E}_j, \quad i = 1,2,3. \tag{11}$$

The angular velocity and the angular acceleration operator can be written as

$$\omega_{ij} = r_{ki} \Omega_{km} r_{mj}, \quad \varepsilon_{ij} = r_{ki} \mathrm{E}_{km} r_{mj}, \quad i = 1,2,3. \tag{12}$$

Let us now consider one finite element of a solid elastic body. This finite element will participate in the general motion of the solid. A method to determine the motion equations of this finite element is to use the Lagrange equations. To apply this method, a first step is to calculate the Lagrangian for this finite element. So it is necessary to determine the kinetic energy, the internal energy and the external work of the concentrated and distributed forces. We assume that the rigid motion of the solid is known and is not influenced by the elastic deformation of the elements. It results that the velocities and accelerations field for the solid are known.

Consider an arbitrary finite element that is refer to a local reference system $Oxyz$, participating to the general three-dimensional rigid motion of finite element finite (Fig.1). The field of velocities and the accelerations for the system is considered being known (that means that the velocity and the acceleration of the origin of the local coordinate system are known). The angular velocity and the angular acceleration of the local coordinate system is too considered as being known.

The displacement δ (u,v,w) of an arbitrary point $M(x,y,z)$ can be written, using the shape functions $\mathbf{N}_{ij}$ and the vector δ_e of the nodal displacements, in the local coordinate system

$$u = \delta_1 = N_{1j} \delta_{e,j}, v = \delta_2 = N_{2j} \delta_{e,j}, \ w = \delta_3 = N_{3j} \delta_{e,j}, \quad j = \overline{1,p} \, , \tag{13}$$

or

$$\boldsymbol{\delta}_i = N_{ij} \delta_{e,j}, \quad i = 1,2,3, \quad j = \overline{1,p} \ . \tag{14}$$

where p is the number of DOF (degrees of freedom) of the element.

The kinetic energy of an element, due to the translation is

$$E_{ct}=\frac{1}{2}\int_0^L \rho \dot{X}'_k \dot{X}'_k\, dV=$$

$$=\frac{1}{2}\int_0^L \rho\left(\dot{X}_{ko}+\dot{r}_{k1}x_1+\dot{r}_{ki}N_{ij}\delta_{eL,j}+r_{ki}N_{ij}\dot{\delta}_{eL,j}\right)\left(\dot{X}_{ko}+\dot{r}_{k1}x_1+\dot{r}_{ki}N_{ij}\delta_{eL,j}+r_{ki}N_{ij}\dot{\delta}_{eL,j}\right)\mathrm{d}V\,. \tag{15}$$

The total internal energy can be determined with the relation

$$E_p=\frac{1}{2}\boldsymbol{\delta}_{e,i}k_{e,ij}\,\boldsymbol{\delta}_{e,j}\,. \tag{16}$$

The concentrated loads, with the local components $\mathbf{q}_{e,i}$, applied in the nodes, cause an external work

$$W^c=\mathbf{q}_{e,i}\boldsymbol{\delta}_{e,i}\,. \tag{17}$$

The external work of the distributed loads is

$$W^d=\int_0^L\left(p_1\delta_1+p_2\delta_2+p_3\delta+m_1\alpha+m_2\beta+m_3\gamma\right)\mathrm{d}x=$$

$$=\left(\int_0^L p_iN_{ij}dx\right)\delta_{e,j}+\left(\int_0^L m_iN_{(i+3),j}dx\right)\delta_{e,j}=q^*_{e,j}\delta_{e,j}\,,\quad i=1,2,3,\quad j=\overline{1,p}\,, \tag{18}$$

where is used the notation

$$q^*_{e,j}=\left(\int_0^L p_iN_{ij}dx\right)\delta_{e,j}+\left(\int_0^L m_iN_{(i+3),j}dx\right)\delta_{e,j}\,,\quad i=1,2,3,\quad j=\overline{1,p}\,. \tag{19}$$

The Lagrangian of the element is [14], [15], [25], [26], [29] - [32]

$$L=E_c-E_p+W^d+W^c\,. \tag{20}$$

The Lagrange’s equations are

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\delta}_{e,i}}-\frac{\partial L}{\partial\delta_{e,i}}=0\,. \tag{21}$$

3. ONE DIMENSIONAL FINITE ELEMENT

Let us consider a finite one-dimensional truss element. (Fig.1). To obtain the motion equations using Lagrange’s equations, the first step is to write the Lagrangian for the element. (which is able to have traction-compression, torsion and bending). To do this it shall be computed the kinetic energy of the considered finite element, the internal energy and the external work of the distributed and concentrated loads.

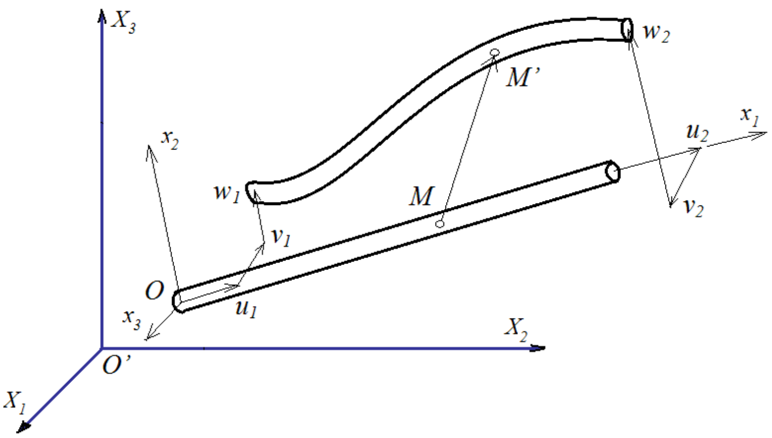

Fig. 1. One-dimensional finite element.

Consider now a truss finite element, having at ends the nodes numbered *i* and *j*. The finite element is referring to a local reference system *Oxyz* , participating to the general three-dimensional rigid motion of the truss (Fig.1). It is considered that the velocity and the acceleration of the origin of the local coordinate reference and the angular velocity and the angular acceleration of the local coordinate system is considered as being known.

In the following we will follow the demonstration presented in the paper [43]. The displacement $\boldsymbol{\delta}\,(u,v,w)$ of an arbitrary point M chosen at a distance x from the left end of the bar can be written, using the shape functions $\mathbf{N}_{ij}$ and the vector of the nodal displacements, in the local coordinate system

$$u = \delta_1 = N_{1j}\delta_{e,j} \quad ; \quad v = \delta_2 = N_{2j}\delta_{e,j}, \quad w = \delta = N_{3j}\delta_{e,j}, \quad j = \overline{1,12} \ , \qquad (22)$$

or:

$$\boldsymbol{\delta}_i = N_{ij}\delta_{e,j}, \quad i = 1,2,3, \quad j = \overline{1,12} \ . \qquad (23)$$

where the nodal displacements vector of the finite element numbered e , δ_e, is

$$\delta_e^T = \left[\delta_1^{(1)}\ \delta_2^{(1)}\ \delta_3^{(1)}\ L\alpha^{(1)}\ L\beta^{(1)}\ L\gamma^{(1)} \vdots \delta_1^{(2)}\ \delta_2^{(2)}\ \delta_3^{(2)}\ L\alpha^{(2)}\ L\beta^{(2)}\ L\gamma^{(2)}\right]. \qquad (24)$$

The entries of this vector are

- $\delta_1^{(1)} = u_1$, $\delta_2^{(1)} = v_1$, $\delta_3^{(1)} = w_1$, the displacement of the left end of the truss along the three directions;
- $\delta_1^{(2)} = u_2$, $\delta_2^{(2)} = v_2$, $\delta_3^{(2)} = w_3$, the displacement of the right end of the truss along the three directions;
- $\alpha^{(1)} = \alpha_1$, $\beta^{(1)} = \beta_1$, $\gamma^{(1)} = \gamma_1$, the rotations of the left end section around the three axes;
- $\alpha^{(2)} = \alpha_1$, $\beta^{(2)} = \beta_2$, $\gamma^{(2)} = \gamma_2$, the rotations of the right end section around the three axes.

The lines of the shape functions matrix $\mathbf{N}$ correspond to the displacements *u*, *v* and *w* and are named $N_{(u)} = N_{(1)}, N_{(v)} = N_{(2)}$ and $N_{(w)} = N_{(3)}$

$$\mathbf{N} = \begin{bmatrix} N_{(u)} \\ N_{(v)} \\ N_{(w)} \end{bmatrix} = \begin{bmatrix} N_{(1)} \\ N_{(2)} \\ N_{(3)} \end{bmatrix} = \left[N_{ij} \right], \quad i = 1,2,3, \quad j = \overline{1,\,12}\ . \tag{23}$$

For the rotation α is adopted the relation

$$\alpha = \delta_4 = N_{4i}\delta_{e,i} \qquad i = \overline{1,12}\,. \tag{24}$$

usually the same shape function as for the axial deformation. The rotations of the transversal section are used the equations well known from the continuous mechanics [23]

$$\beta = \delta_5 = -\frac{\mathrm{d}w}{\mathrm{d}x} \quad \text{and} \quad \gamma = \delta_6 = \frac{\mathrm{d}v}{\mathrm{d}x}\ . \tag{25}$$

and can be expressed as follows

$$\beta = -\frac{d}{dx}\left(\mathbf{N}_{3i}\,\boldsymbol{\delta}_{e,i}\right) = -\mathbf{N}'_{3i}\boldsymbol{\delta}_{e,i} = \mathbf{N}_{5i}\boldsymbol{\delta}_{e,i}\,, \quad \gamma = \frac{d}{dx}\left(\mathbf{N}_{2i}\boldsymbol{\delta}_{e,i}\right) = \mathbf{N}'_{2i}\boldsymbol{\delta}_{e,i} = \mathbf{N}_{6i}\boldsymbol{\delta}_{e,i}\ , \tag{26}$$

$$\boldsymbol{\delta}_i = N_{ij}\delta_{e,j} \quad i = 4,5,6\ . \tag{27}$$

After deformation of the element the displacement of the point $M(x_1, x_2, x_3)$ becomes $M'(x_1', x_2', x_3')$

$$x'_1 = x_1 + u = x_1 + \delta_1, \quad x'_2 = v = \delta_2, \quad x'_3 = w = \delta_3\ , \tag{28}$$

or, with respect to the global coordinate system

$$X'_1 = X_1 + r_{1i}\delta_i = X_{1o} + r_{11}x_1 + r_{1i}\delta_i = X_{1o} + r_{11}x_1 + r_{1i}N_{ij}\delta_{e,j}\,,$$
$$X'_2 = X_2 + r_{2i}\delta_i = X_{2o} + r_{21}x_1 + r_{2i}\delta_i = X_{2o} + r_{21}x_1 + r_{2i}N_{ij}\delta_{e,j}\,,$$
$$X'_3 = X_3 + r_{3i}\delta_i = X_{3o} + r_{31}x_1 + r_{3i}\delta_i = X_{3o} + r_{31}x_1 + r_{3i}N_{ij}\delta_{e,j}\,,$$
$$i = 1,2,3,\ j = \overline{1,\,12}\ , \tag{29}$$

or

$$X'_k = X_{ko} + \alpha_{k1}x_1 + \alpha_{ki}N_{ij}\delta_{eL,j}, \quad k = \overline{1,3}, \tag{30}$$

The velocity is

$$\dot{X}'_k = \dot{X}_{ko} + \dot{r}_{k1}x_1 + \dot{r}_{ki}N_{ij}\delta_{eL,j} + r_{ki}N_{ij}\dot{\delta}_{eL,j}, \quad k = \overline{1,3}\,. \tag{31}$$

The kinetic energy due to the translation is

$$E_{ct} = \frac{1}{2}\int_0^L \rho A \dot{X}'_k\,\dot{X}'_k\,dx_1 =$$

$$= \frac{1}{2}\int_0^L \rho A\left(\dot{X}_{ko} + \dot{r}_{k1}x_1 + \dot{r}_{ki}N_{ij}\delta_{eL,j} + r_{ki}N_{ij}\dot{\delta}_{eL,j}\right)\left(\dot{X}_{ko} + \dot{r}_{k1}x_1 + \dot{r}_{ki}N_{ij}\delta_{eL,j} + r_{ki}N_{ij}\dot{\delta}_{eL,j}\right)dx_1 \tag{32}$$

The kinetic energy of the element *dm*, due to the rotation is

$$E_{cr} = \frac{1}{2}\int_0^L \rho\, \boldsymbol{\omega}'_i\, \mathbf{I}_{ij}\, \boldsymbol{\omega}'_j\, \mathrm{d}x\,. \tag{33}$$

The infinitesimal element *dm* has the angular velocity $\{\boldsymbol{\omega}'_L\}$ with the components

$$\boldsymbol{\omega}'_1 = \omega_1 + \dot{\alpha}, \quad \boldsymbol{\omega}'_2 = \omega_2 + \dot{\beta}, \quad \boldsymbol{\omega}'_3 = \omega_3 + \dot{\gamma}\,, \tag{34}$$

or, taking into account the shape functions (22)

$$\omega'_i = \omega_i + N_{i+3,j}\dot{\delta}_{e,j}\;\;. \tag{35}$$

From the rel.(24) and (26) we obtain:

$$\dot{\alpha} = N_{4i}\,\dot{\delta}_{e,i}\,,\;\; \dot{\beta} = \mathbf{N}_{5i}\,\dot{\boldsymbol{\delta}}_{e,i}\,,\;\; \dot{\gamma} = \mathbf{N}_{6i}\,\dot{\boldsymbol{\delta}}_{e,i}. \tag{36}$$

The values $\omega_1,\ \omega_2,\ \omega_3$ represent components of the angular velocity vector related to the local (mobile) coordinate system.

The inertia matrix is

$$\mathbf{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & -I_{yz} \\ 0 & -I_{yz} & I_{zz} \end{bmatrix}. \tag{37}$$

The values I_{yy} and I_{zz} represent the moments of inertia of the bar cross section about the axis Oy and Oz respectively. The coordinate system has the origin in the mass center of the element $dm = \rho A dx$ (ρ - density), I_{yz} is the centrifugal moment of inertia and I_{xx} is the inertia moment about the axis Ox. Since we have chosen y and z as principal directions of inertia $I_{yz} = 0$, the matrix of moments of inertia become

$$\mathbf{I} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}, \tag{38}$$

where, for the sake of simplicity the notations $I_{xx} = I_x$, $I_{yy} = I_y$, $I_{zz} = I_z$ are made.

In the following shall be calculated the internal energy stored in the truss:

$$E_{pb} = \frac{1}{2}\int_0^L \left[EI_y \left(\frac{d^2 w}{dx^2} \right)^2 + EI_z \left(\frac{d^2 v}{dx^2} \right)^2 \right] \mathrm{d}x = \frac{1}{2}\int_0^L \left[EI_y \beta'^2 + EI_z \gamma'^2 \right] \mathrm{d}x\;\;. \tag{39}$$

If we introduce the expression of *w* si *v* [22] it obtains

$$E_{pb} = \frac{1}{2}\boldsymbol{\delta}_{e,i}\left[\int_0^L \left(EI_y \mathbf{N}''_{3i}\mathbf{N}''_{3j} + EI_z \mathbf{N}''^{\,T}_{2i}\mathbf{N}''_{2j} \right) dx \right]\boldsymbol{\delta}_{e,j} = \frac{1}{2}\boldsymbol{\delta}_{e,i} k_{b,ij} \boldsymbol{\delta}_{e,j}\,, \tag{40}$$

where

$$k_{b,ij} = \int_0^L \left(EI_y \mathbf{N}''_{3i}\mathbf{N}''_{3j} + EI_z \mathbf{N}''^{\,T}_{2i}\mathbf{N}''_{2j} \right) \mathrm{d}x\;\;. \tag{41}$$

The energy caused by the axial deformation is

$$E_{pa} = \frac{1}{2}\int_0^L EA\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \mathrm{d}x = \frac{1}{2}\boldsymbol{\delta}_{eL,i}\left(\int_0^L \mathbf{N}'_{1i}\,\mathbf{N}'_{1j}\,EA\mathrm{d}x\right)\boldsymbol{\delta}_{eL,j} = \frac{1}{2}\boldsymbol{\delta}_{e,i}k_{a,ij}\boldsymbol{\delta}_{e,j}\,, \tag{42}$$

with

$$k_{a,ij} = \int_0^L \mathbf{N}'_{1i}\,\mathbf{N}'_{1j}\,EA\mathrm{d}x\ . \tag{43}$$

The deformation energy due to torsion is

$$E_{pt} = \frac{1}{2}\int_0^L GI_x\left(\frac{\mathrm{d}\alpha}{\mathrm{d}x}\right)^2 \mathrm{d}x = \frac{1}{2}\boldsymbol{\delta}_{eL,i}\left(\int_0^L \mathbf{N}'_{4i}{}^T\mathbf{N}'_{4j}\,GI_x\mathrm{d}x\right)\boldsymbol{\delta}_{eL,j} = \frac{1}{2}\boldsymbol{\delta}_{e,i}k_{t,ij}\,\boldsymbol{\delta}_{e,j}\,, \tag{44}$$

with

$$k_{t,ij} = \int_0^L \mathbf{N}'_{4i}\,\mathbf{N}'_{4j}\,GI_x\mathrm{d}x\ . \tag{45}$$

It can be considered the effects of an axial load P_{tot} existing in an axial section of the bar, that gives the following energy if, in a first approximation, the axial deformations are neglected

$$E_a = \frac{1}{2}\int_0^L P_{tot}\left[\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right]\mathrm{d}x\ , \tag{46}$$

where P_{tot} represents the axial force in the bar cross section at the distance x. We consider that the force components acting at the right bar end are, in the local coordinate system: P_x, $P_y=0$, $P_z=0$. With these assumptions, we will determine the components of the inertia forces acting upon the portion of the bar between x and L (Fig.2).

The current point of the bar, with the abscissa x has the acceleration

$$a_{G,1} = a_{o1,G} + (r_{12}\varepsilon_3 - r_{13}\varepsilon_2)\,x - \left[r_{11}(\omega_2^2+\omega_3^2) - r_{12}\omega_1\omega_2 - r_{13}\omega_1\omega_3\right]x\,. \tag{47}$$

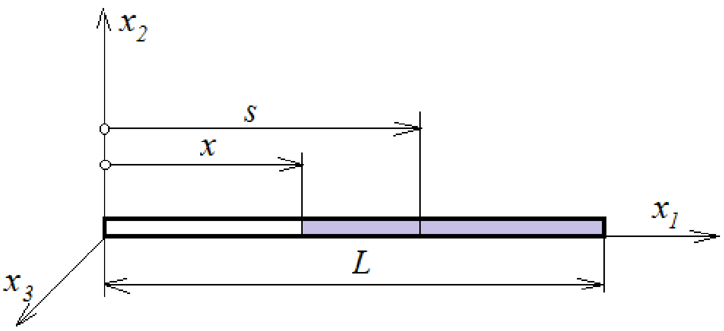

Fig. 2. Determination of the axial inertia force.

The inertia force is given by

$$\mathbf{F}_{i,1} = -\int_x^L \mathbf{a}_{G,1}dm = -\int_x^L \mathbf{a}_{ox,G}\rho A ds - \int_x^L (r_{12}\varepsilon_3 - r_{13}\varepsilon_2)\,s\rho A\mathrm{d}s + \int_x^L \left[r_{11}(\omega_2^2+\omega_3^2) - r_{12}\omega_1\omega_2 - r_{13}\omega_1\omega_3\right]s\rho A\mathrm{d}s =$$

$$= -\ddot{X}_o \rho A(L-x) - \frac{1}{2}(r_{12}\varepsilon_3 - r_{13}\varepsilon_2)\rho A(L^2 - x^2) + \frac{1}{2}\left[r_{11}(\omega_2^2+\omega_3^2) - r_{12}\omega_1\omega_2 - r_{13}\omega_1\omega_3\right](L^2 - x^2)\rho A/. \tag{48}$$

We denote with

$$\mu = -\ddot{X}_o \rho A L - \frac{1}{2}(r_{12}\varepsilon_3 - r_{13}\varepsilon_2)\rho A L^2 + \frac{1}{2}\left[r_{11}(\omega_2^2+\omega_3^2) - r_{12}\omega_1\omega_2 - r_{13}\omega_1\omega_3\right]\rho A L^2,$$

$$\lambda = \ddot{X}_o \rho A \quad ; \quad \nu = \frac{1}{2}(r_{12}\varepsilon_3 - r_{13}\varepsilon_2)\rho A - \frac{1}{2}\left[r_{11}(\omega_2^2+\omega_3^2) - r_{12}\omega_1\omega_2 - r_{13}\omega_1\omega_3\right]\rho A. \tag{49}$$

The internal energy due to inertia of the mass of bar is

$$E_a = \frac{1}{2}\boldsymbol{\delta}_{eL,i}\left[\int_0^L \left(P_x + \mu_x + \lambda_x x + \nu_x x^2\right)\left(\mathbf{N}_{3i}^*\mathbf{N}_{3j}^* + \mathbf{N}_{2i}^*\mathbf{N}_{2j}^*\right)dx_1\right]\boldsymbol{\delta}_{eL,j} =$$

$$= \frac{1}{2}\boldsymbol{\delta}_{eL,i}\left[\int_0^L \left(P_x + \mu_x + \lambda_x x + \nu_x x^2\right)\left(\mathbf{N}_{2i}\mathbf{N}_{2j} + \mathbf{N}_{3i}\mathbf{N}_{3j}\right)dx_1\right]\boldsymbol{\delta}_{eL,j} = \frac{1}{2}\boldsymbol{\delta}_{e,i}k_{ij}^G\,\boldsymbol{\delta}_{e,j}\ , \tag{50}$$

where

$$k_{ij}^G = \int_0^L \left(P_x + \mu_x + \lambda_x x + \nu_x x^2\right)\left(\mathbf{N}_{2i}\mathbf{N}_{2j} + \mathbf{N}_{3i}\mathbf{N}_{3j}\right)\mathrm{d}x_1\ . \tag{51}$$

The total internal energy is

$$E_p = \frac{1}{2}\boldsymbol{\delta}_{e,i}\left(\mathbf{k}_{b,ij} + \mathbf{k}_{a,ij} + \mathbf{k}_{t,ij} + \mathbf{k}_{ij}^G\right)\boldsymbol{\delta}_{e,j} = \frac{1}{2}\boldsymbol{\delta}_{e,i}k_{e,ij}\,\boldsymbol{\delta}_{e,j}. \tag{52}$$

where the notation

$$k_{e,ij} = \mathbf{k}_{b,ij} + \mathbf{k}_{a,ij} + \mathbf{k}_{t,ij} + \mathbf{k}_{ij}^G, \tag{53}$$

is used. The external work of the concentrated loads with the local components $\mathbf{q}_{e,i}$ applied in the nodes is

$$W^c = \mathbf{q}_{e,i}\boldsymbol{\delta}_{e,i}. \tag{54}$$

The external work of the distributed loads is

$$W^d = \int_0^L \left(p_1 u + p_2 v + p_3 w + m_1\alpha + m_2\beta + m_3\gamma\right)\mathrm{d}x =$$

$$= \left(\int_0^L p_i N_{ij} dx\right)\delta_{e,j} + \left(\int_0^L m_i N_{(i+3),j} dx\right)\delta_{e,j} = q_{e,j}^*\delta_{e,j}\ , \quad i = 1,2,3, \quad j = \overline{1,12}\ , \tag{55}$$

where the notation

$$q_{e,j}^* = \left(\int_0^L p_i N_{ij} dx\right)\delta_{e,j} + \left(\int_0^L m_i N_{(i+3),j} dx\right)\delta_{e,j}\ , \quad i = 1,2,3, \quad j = \overline{1,12}. \tag{56}$$

is used. The Lagrangian of the element becomes

$$L = E_c - E_p + W^d + W^c. \tag{57}$$

or, taking into account the rel. (32),(33),(40),(42),(44),(50),(52),(54)-(56)

$$L=\frac{1}{2}\int_0^L \rho A\left(\dot{X}_{ko}+\dot{r}_{k1}x_1+\dot{r}_{ki}N_{ij}\delta_{eL,j}+r_{ki}N_{ij}\dot{\delta}_{eL,j}\right)\left(\dot{X}_{ko}+\dot{r}_{k1}x_1+\dot{r}_{ki}N_{ij}\delta_{eL,j}+r_{ki}N_{ij}\dot{\delta}_{eL,j}\right)dx_1-$$

$$-\frac{1}{2}\boldsymbol{\delta}_{eL,i}k_{e,ij}\,\boldsymbol{\delta}_{eL,j}+q^*_{eL,j}\delta_{eL,j}+\mathbf{q}_{eL,i}\boldsymbol{\delta}_{eL,i}\ . \quad (58)$$

Applying the Lagrange's equations [21], written in the form

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\delta}_{e,i}}-\frac{\partial L}{\partial\delta_{e,i}}=0\,. \quad (59)$$

the motion equations written in the local coordinate system, for one-dimensional finite element, take the form

$$\mathbf{m}_{e,ij}\ddot{\boldsymbol{\delta}}_{eL,j}+2\mathbf{c}^{\omega}_{e,ij}\dot{\boldsymbol{\delta}}_{eL,j}+\left(\mathbf{k}_{e,ij}+\mathbf{k}^{\varepsilon}_{e,ij}+\mathbf{k}^{\omega^2}_{e,ij}\right)\boldsymbol{\delta}_{eL,j}=$$

$$=\mathbf{q}_{e,i}+\mathbf{q}^*_{e.i}-\mathbf{q}^{\varepsilon}_{e,i}-\mathbf{q}^{\omega^2}_{e,i}-\mathbf{m}^{\varepsilon}_{e,ik}\mathbf{I}_{kj}\,\boldsymbol{\varepsilon}_{L,j}-\mathbf{m}^{o}_{e,ij}\ddot{x}_{jo}\,, \quad (60)$$

where

$$\mathbf{m}_{e,ij}=\mathbf{m}_{t,ij}+\mathbf{m}_{r,ij}\ ,\quad \mathbf{m}_{t,ij}=\int_0^L\rho A\mathbf{N}_{ki}\mathbf{N}_{kj}dx_1\,,\quad i,j=\overline{1,12},\quad k=1,2,3\ .;$$

$$\mathbf{m}_{r,ij}=\int_0^L\rho\mathbf{N}^*_{k+3,i}\mathbf{I}_{kl}\,\mathbf{N}^*_{l+3,j}dx\,,\quad k,l=1,2,3,\quad i,j=\overline{1,12}\quad;\quad \mathbf{c}_{e,ij}(\omega)=\int_0^L\mathbf{N}_{ki}\,\boldsymbol{\omega}_{L,km}\mathbf{N}_{mj}\rho A dx_1\,,$$

$$\mathbf{k}^{\varepsilon}_{e,ij}=\int_0^L\mathbf{N}_{ki}\boldsymbol{\varepsilon}_{L,km}\mathbf{N}_{mj}\rho A dx_1\,,\quad \mathbf{k}^{\omega^2}_{e,ij}=\int_0^L\mathbf{N}_{ki}\omega_{L,km}\boldsymbol{\omega}_{L,ml}\mathbf{N}_{lj}\rho A dx_1\,,$$

$$\mathbf{m}^{o}_{e,ij}=\int_0^L\rho A\mathbf{N}_{ji}dx\,,\quad i=\overline{1,12},\quad j=1,2,3\,,\quad \mathbf{m}^{\varepsilon}_{e,ij}=\int_0^L\mathbf{N}_{j,i+3}dx_1\,,\quad \ddot{x}_{jo}=\mathbf{R}^T\ddot{X}_{jo}\,,$$

$$\mathbf{m}^{x}_{e,ij}=\int_0^L\mathbf{N}_{ji}x\rho A dx\,,\quad i,j=1,2,3.$$

An extended presentation of the motion equations are

$$\left[\int_0^L\rho A\mathbf{N}_{ik}\mathbf{N}_{jk}dx_1+\int_0^L\rho\mathbf{N}^*_{ki}\mathbf{I}_{kl}\,\mathbf{N}^*_{lj}dx\right]\ddot{\delta}_{eL,j}+2\left[\int_0^L\mathbf{N}_{ki}\,\boldsymbol{\omega}_{L,km}\mathbf{N}_{mj}\rho A dx_1\right]\dot{\delta}_{eL,j}+$$

$$+\left[k_{e,ij}+\int_0^L\mathbf{N}_{ki}\boldsymbol{\varepsilon}_{L,km}\mathbf{N}_{mj}\rho A dx_1+\int_0^L\mathbf{N}_{ki}\omega_{L,km}\boldsymbol{\omega}_{L,ml}\mathbf{N}_{lj}\rho A dx_1\right]\delta_{eL,j}=$$

$$=\mathbf{q}_{e,i}+\mathbf{q}^*_{e.i}-\mathbf{q}^{\varepsilon}_{e,i}-\mathbf{q}^{\omega^2}_{e,i}-\left(\int_0^L\mathbf{N}_{j,i+3}dx_1\right)\mathbf{I}_{kj}\,\boldsymbol{\varepsilon}_{L,j}-\left(\int_0^L\rho A\mathbf{N}_{ji}dx\right)\ddot{x}_{jo}\ . \quad (61)$$

With the notation above mentioned the concise form (60) is obtained. If the element is considered to have a constant cross-section is possible to obtain easy the results and the coefficient after polynomial integrations.

4. THREE DIMENSIONAL FINITE ELEMENT

We shall note with $\overline{v}_{o}(\dot{X}_o,\dot{Y}_o,\dot{Z}_o)$ the velocity and with $\overline{a}_{o}(\ddot{X}_o,\ddot{Y}_o,\ddot{Z}_o)$ the acceleration of the origin of the local coordinate relative to the global coordinate system OXYZ, to which the motion of the whole system will relate. We shall note the angular velocity with $\overline{\omega}(\omega_x,\omega_y,\omega_z)$ and with $\overline{\varepsilon}(\varepsilon_x,\varepsilon_y,\varepsilon_z)$ the angular acceleration. The multi-body system consisting of several solids, these vectors will be different for each solid composing the system. The transformation of a vector from the local system of coordinates into the global system of coordinates occurs by means of a matrix of rotation $\boldsymbol{R}$. In this section we will follow the results presented in [40].

The displacement $\boldsymbol{\delta}(u,v,w)$ of an arbitrary point M chosen at a distance x from the left end of the bar can be written, using the shape functions $\mathbf{N}_{ij}$ and the vector of the nodal displacements, in the local coordinate system

$$u=\delta_1=N_{1j}\delta_{e,j}, v=\delta_2=N_{2j}\delta_{e,j},\quad w=\delta=N_{3j}\delta_{e,j},\quad j=\overline{1,12}, \tag{62}$$

or

$$\boldsymbol{\delta}_i=N_{ij}\delta_{e,j}\quad i=1,2,3;\quad j=\overline{1,12}\ . \tag{63}$$

where the nodal displacements vector of the finite element numbered e , δ_e, is:

$$\delta_e^T=\left[u_1\ v_1\ w_1\ u_2\ v_2\ w_2\right]\ . \tag{64}$$

Here u,v and w are the displacement of the current point of the element along the three axis Ox,Oy,Oz.

$$\mathbf{N}=\begin{bmatrix}N_{(u)}\\N_{(v)}\\N_{(w)}\end{bmatrix}=\begin{bmatrix}N_{(1)}\\N_{(2)}\\N_{(3)}\end{bmatrix}=\left[N_{ij}\right],\quad i=1,2,3;\quad j=\overline{1,\ p}\ . \tag{65}$$

The point $M(x_1,x_2,x_3)$ becomes, after deformation $M'(x_1',x_2',x_3')$

$$x'_1=x_1+u=x_1+\delta_1;\quad x'_2=x_2+v=x_2+\delta_2;\quad x'_3=x_3+w=x_3+\delta_3, \tag{66}$$

or, with respect to the global coordinate system

$$\begin{aligned}X'_1&=X_1+r_{1i}\delta_i=X_{1o}+r_{1i}x_i+r_{1i}\delta_i=X_{1o}+r_{1i}x_i+r_{1i}N_{ij}\delta_{e,j}\ ,\\X'_2&=X_2+r_{2i}\delta_i=X_{2o}+r_{2i}x_i+r_{2i}\delta_i=X_{2o}+r_{2i}x_i+r_{2i}N_{ij}\delta_{e,j},\\X'_3&=X_3+r_{3i}\delta_i=X_{3o}+r_{3i}x_i+r_{3i}\delta_i=X_{3o}+r_{3i}x_i+r_{3i}N_{ij}\delta_{e,j},\\&i=1,2,3;\quad j=\overline{1,\,12}\ ,\end{aligned} \tag{67}$$

or

$$X'_k=X_{ko}+r_{ki}x_i+r_{ki}N_{ij}\delta_{eL,j},\quad k=\overline{1,3}\ . \tag{68}$$

The velocity is

$$\dot{X}'_k = \dot{X}_{ko} + \dot{r}_{ki}x_i + \dot{r}_{ki}N_{ij}\delta_{eL,j} + r_{ki}N_{ij}\dot{\delta}_{eL,j}, \quad k=\overline{1,3}. \tag{69}$$

The kinetic energy due to the translation is, in this case

$$E_{ct} = \frac{1}{2}\int_0^L \rho \dot{X}'_k \, \dot{X}'_k \,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 =$$

$$= \frac{1}{2}\int_0^L \rho\left(\dot{X}_{ko} + \dot{r}_{ki}x_i + \dot{r}_{ki}N_{ij}\delta_{eL,j} + r_{ki}N_{ij}\dot{\delta}_{eL,j}\right)\left(\dot{X}_{ko} + \dot{r}_{ki}x_i + \dot{r}_{ki}N_{ij}\delta_{eL,j} + r_{ki}N_{ij}\dot{\delta}_{eL,j}\right)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3. \tag{70}$$

The equations of motion shall be obtained in the local coordinate system but they can also be obtained in the global coordinate system. For this purpose, the equations of Lagrange shall be used. The potential energy (internal work) of the elemnt is:

$$E_p = \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}\mathrm{d}V\ . \tag{71}$$

where σ represents the stress tensor and ε the strains tensor.

The Hooke law can be written as follows

$$\sigma_{ij} = D_{ik}\varepsilon_{kj}\ . \tag{72}$$

The differential relations which link the strains to the finite deformations can be expressed in a concise form:

$$\varepsilon_{kj} = a_{km}\delta_{mj}\,, \tag{73}$$

where a represents the differentiation operator [44].

Using the rel. (72) and (73) it results the strains energy

$$E_p = \frac{1}{2}\int_V \delta_{\mathrm{e,ik}}\ k_{e,im}\delta_{e,ml}dV \quad , \tag{74}$$

where $k_{e,im}$ is the stiffness matrix

$$k_{e,im} = \int_V N_{ji}a_{kj}D_{lk}a_{\ln}N_{nm}dV\ . \tag{75}$$

For the considered element the Lagrangian will be:

$$L = E_c - E_p + W + W^c \quad . \tag{76}$$

Applying the equations of Lagrange

$$\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\delta}_e}\right\} - \left\{\frac{\partial L}{\partial \delta_e}\right\} = 0 \tag{77}$$

The motion equations are obtained in the form

$$\mathbf{m}_{e,ij}\ddot{\boldsymbol{\delta}}_{eL,j} + 2\mathbf{c}^{\omega}_{e,ij}\dot{\boldsymbol{\delta}}_{eL,j} + \left(\mathbf{k}_{e,ij} + \mathbf{k}^{\varepsilon}_{e,ij} + \mathbf{k}^{\omega^2}_{e,ij}\right)\boldsymbol{\delta}_{eL,j} =$$

$$= \mathbf{q}_{e,i} + \mathbf{q}^{*}_{e.i} - \mathbf{q}^{\varepsilon}_{e,i} - \mathbf{q}^{\omega^2}_{e,i} - \mathbf{m}^{o}_{e,ij}\ddot{x}_{jo} \quad .$$

(78)

By $\left\{\frac{\partial L}{\partial \delta_e}\right\}$ we understand

$$\left\{\frac{\partial L}{\partial X}\right\} = \begin{Bmatrix} \frac{\partial L}{\partial x_1} \\ \frac{\partial L}{\partial x_2} \\ \vdots \\ \frac{\partial L}{\partial x_n} \end{Bmatrix} \quad \text{where } \{X\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} . \tag{79}$$

The matrix coefficients are determined choosing the shape functions and the nodal coordinates of a point.

5. ANALYSIS OF THE MOTION EQUATIONS

The motion equations for one single finite element finit for an one-dimensional, two-dimensional or three-dimesnional element is

$$\mathbf{m}_{e,ij}\ddot{\boldsymbol{\delta}}_{eL,j} + 2\mathbf{c}^{\omega}_{e,ij}\dot{\boldsymbol{\delta}}_{eL,j} + \left(\mathbf{k}_{e,ij} + \mathbf{k}^{\varepsilon}_{e,ij} + \mathbf{k}^{\omega^2}_{e,ij}\right)\boldsymbol{\delta}_{eL,j} =$$

$$= \mathbf{q}_{e,i} + \mathbf{q}^{*}_{e.i} - \mathbf{q}^{\varepsilon}_{e,i} - \mathbf{q}^{\omega^2}_{e,i} / -\mathbf{m}^{\varepsilon}_{e,ik}\mathbf{I}_{kj}\,\boldsymbol{\varepsilon}_{L,j} / -\mathbf{m}^{o}_{e,ij}\ddot{x}_{jo} \ .$$

Matrix coefficients involved in these equations are:

- The inertia tensor $\mathbf{m}_{e,ij}$, a symmetrical one;
- The damping tensor $\mathbf{c}_{e,ij}$ is a skew symmetric tensor and represent accelerations due to relative motions of nodal displacements with respect to the mobile reference co-ordinate system (Coriolis type acceleration);
- The rigidity tensor $\mathbf{k}_{e,ij}$ is symmetric too.
- The change of the rigidity due to the relative motion of the elastic element $\mathbf{k}^{\varepsilon}_{e,ij} + \mathbf{k}^{\omega^2}_{e,ij}$. These terms can determine singularities of the total rigidity matrix and, as a consequence, loss of the stability;
- The vector of the generalized loads contains, beside external (concentrated and distributed) loads, terms due to inertia of finite elements being in rigid motion.

The system of differential equations obtained is nonlinear (for some structure of MBS can be strong non-linear), the matrix coefficients of the system depending on time. The method mostly used to solve a such system is that of linearizing these equations considering the tensor coefficients as being constant for very short time intervals (rigid motion freezing). In this case a system of differential equations with constant coefficients is obtained, the solving procedures for these equations being well known. Nonlinear aspects are due to the “conservative” damping caused by the skew symmetric tensor $\mathbf{c}_{ij}$ and by the modification of the rigidity matrix due to relative motions.

6. CONCLUSIONS

Researches dedicated to multibody systems with elastic elements takes place over a period of five decades (even if the term multibody system began later). The main outcomes of the theoretical and numerical problems implied by this approach have been identified and set out during this period. A number of types of finite elements have been proposed to study the problem and a number of applications - usually very simple - have been solved. The modeling of systems and the achievement of motion equations is a stage already studied by researchers and reinforced by numerous published articles. However, the practical application of the calculation methods was made mostly on simple systems with a low degree of complexity. The main problem with unpleasant practical effects is the loss of stability and the occurrence of resonance phenomena. Changing element lengths due to elasticity and vibrations may also result in poor operation of a system or machine and may make it unnecessary for the purpose for which it was designed and built. A dificulty remains the high degree of complexity of solving a multibody system, which has a rigid motion, over which a movement determined by the elasticity of the elements overlaps. The Coriolis effects and complex relative motion in multibody systems modify the classic (well known) motion equations in the case of a finite element analysis. The most important problem, open to the study, is to find ways to make it possible to integrate motion equations and to quickly get conclusions about the behavior of such a system.

REFERENCES

1. BAGCI, C., *Elastodynamic Response of Mechanical Systems using Matrix Exponential Mode Uncoupling and Incremental Forcing Techniques with Finite Element Method. Proceedings of the Sixth* Word Congress on Theory of Machines and Mechanisms, India, p. 472, 1983.
2. BAKR, E.M., SHABANA, A.A., *Geometrically nonlinear analysis of multibody systems. Computers & Structures*, 23(6), 739–751, 1986.
3. BAHGAT, B.M., WILLMERT, K.D., *Finite Element Vibrational Analysis of Planar Mechanisms*. Mech. Mach. Theory, vol.11, p.47, 1976.
4. BLAJER, W., KOŁODZIEJCZYK, K., *Improved DAE formulation for inverse dynamics simulation of cranes*, Multibody Syst Dyn 25, 131–143, 2011.
5. BOSCARIOL, P., GALLINA,P., GASPARETTO, A., GIOVAGNONI, M., SCALERA, L., VIDONI, R., *Evolution of a Dynamic Model for Flexible Multibody Systems*. Advances in Italian Mechanism Science pp 533-541, 2016.
6. CLEGHORN, W.L., FENTON, E.G., TABARROK, K.B., *Finite Element Analysis of High-Speed Flexible Mechanisms*. Mech. Mach. Theory, vol. 16, p. 407, 1981.
7. CHRISTENSEN, E.R., LEE S.W., *Nonlinear finite element modelling of the dynamics of unrestrained flexible structures.* Computers & Structures, 23(6), 819–829, 1986.
8. DEÜ, J-F, GALUCIO, A.C., OHAYON, R., *Dynamic responses of flexible-link mechanisms with passive/active damping treatment.* Computers & Structures, 86(3–5), 258–265, 2008.
9. ERDMAN, A.G., SANDOR, G.N., OAKBERG, A, *A General Method for Kineto-Elastodynamic Analysis and Synthesis of Mechanisms.* Journal of Engineering for Industry. ASME Trans., 1193, 1972.

10. FANGHELLA, P., GALLETTI, C., TORRE, G., *An explicit independent-coordinate formulation for the equations of motion of flexible multibody systems*. Mech. Mach. Theory, 38, 417–437, 2003.
11. GERSTMAYR, J., SCHÖBERL, J., *A 3D Finite Element Method for Flexible Multibody Systems*. Multibody System Dynamics, 15(4), 305-320, 2006.
12. HOU, W., ZHANG, X., *Dynamic analysis of flexible linkage mechanisms under uniform temperature change*. Journal of Sound and Vibration, 319(1–2), 570–592, 2009.
13. IBRAHIM BEGOVIĆ, A., MAMOURI, S., TAYLOR, R.L., CHEN, A.J., *Finite Element Method in Dynamics of Flexible Multibody Systems: Modeling of Holonomic Constraints and Energy Conserving Integration Schemes*, Multibody System Dynamics, 4(2-3), 195-223, 2000.
14. KHANG, N.V., *Kronecker product and a new matrix form of Lagrangian equations with multipliers for constrained multibody systems*, Mechanics Research Communications, 38(4), 294–299, 2011.
15. KHULIEF, Y.A., *On the finite element dynamic analysis of flexible mechanisms*. Computer Methods in Applied Mechanics and Engineering, 97(1), 23-32, 1992.
16. LAUB, T., OBERPEIL STEINER, St., STEINER,W., NACHBAGAUER, K., *The discret adjoint method for parameter identification in multibody system dynamics*. Multibody Syst. Dyn., 42, 397-410, 2018.
17. MAYO, J., DOMÍNGUEZ, J., *Geometrically non-linear formulation of flexible multibody systems in terms of beam elements: Geometric stiffness*. Computers & Structures, 59(6) 1039, 1996.
18. MIDHA, A., ERDMAN, A.G., FROHRIB, D.A., *Finite element approach to mathematical modeling of high-speed elastic linkages*. Mech. Mach. Theory, 13(6), 603–618, 1978.
19. NATH, P.K., GHOSH, A., 1980, *Steady-state response of mechanism with elastic links by finite element methods.* Mech.Mach.Theory, vol. 15, p. 199
20. NEGREAN, I., *Advanced notions in Analytical Dynamics of Systems. Acta Technica Napocensis - Applied Mathematics*, Mechanics and Engineering, 60(4), 2017.
21. NEGREAN, I., *Advanced Equations in Analytical Dynamics of Systems*, Acta Technica Napocensis, Series: Applied Mathematics, Mechanics and Engineering, 60(IV), 503-514, 2017.
22. NETO, M.A., AMBRÓSIO, J.A.C., LEAL, R.P., *Composite materials in flexible multibody systems.* Computer Methods in Applied Mechanics and Engineering, 195(50–51), 6860–6873, 2006.
23. PENNESTRI', E, de FALCO, D, VITA, L., *An Investigation of the Inuence of Pseudoinverse Matrix Calculations on Multibody Dynamics by Means of the Udwadia-Kalaba Formulation*, Journal of Aerospace Engineering, 22(4), 365-372, 2009.
24. PIRAS, G, CLEGHORN, W.L., MILLS, J.K., *Dynamic finite-element analysis of a planar high speed, high-precision parallel manipulator with flexible links*. Mech. Mach. Theory, 40(7), 849–862, 2005.
25. SCHIELEN, W., *The long History of Impact Mechanics, Rolling Contact and Multibody System Dynamics.* Proc. Appl. Math. Mech. 17, 165-166, 2017.
26. SHABANA, A.A., *Flexible Multibody Dynamics: Review of Past and Recent Developments. Multibody* System Dynamics, 1, 189-222, 1997.
27. SHI, Y.M., LI, Z.F., HUA, H.X., FU, Z.F., LIU, T.X., *The Modelling and Vibration Control of Beams with Active Constrained Layer Damping.* Journal of Sound and Vibration, 245(5), 785–800, 2001.
28. SIMEON, B, *On Lagrange multipliers in flexible multibody dynamic*. Computer Methods in Applied Mechanics and Engineering, 195(50-51), 6993-7005, 2006.
29. STAICU, Șt., *Dynamics analysis of a two-module hybrid parallel manipulator*. Revue Roumaine des Sciences Techniques–Série de Mécanique Appliquée, 59(3), 293-303, 2014
30. STAICU, Șt., OCNĂRESCU, C., UNGUREANU, L., *Dynamics modelling of a paralel flight simulator*, Revue Roumaine des Sciences Techniques–Série de Mécanique Appliquée, 58(3), 273-285, 2013.

31. STĂNESCU, N.D., PANDREA, N., *Dynamics of a Rigid with Many Points Situated on Fixed Surfaces by a Multibody Approach*, Romanian Journal of Mechanics, 1(1), 21-30, 2016.
32. SUNG, CK, *An Experimental Study on the Nonlinear Elastic Dynamic Response of Linkage Mechanism.* Mech. Mach. Theory, 21, 121–133, 1986.
33. TEODORESCU, P.P., *Mechanical Systems, Classical Models*. Vol. II, Springer, 2007.
34. THOMPSON, B.S., SUNG, C.K., *A survey of Finite Element Techniques for Mechanism Design*. Mech. Mach. Theory, 21(4), 351–359, 1986.
35. VLASE, S, *Contributions to the elastodynamic analysis of the mechanisms with the finite element method.* Ph.D., TRANSILVANIA University, 1989.
36. VLASE, S, *Dynamical Response of a Multibody System with Flexible Elements with a General Three Dimensional Motion*, Romanian Journal of Physics, 57(3-4), 676–693, 2012.
37. VLASE, S, *A Method of Eliminating Lagrangean Multipliers from the Equations of Motion of Interconnected Mechanical Systems*. Journal of applied Mechanics, ASME Transactions, 54(1), 1987.
38. VLASE, S, *Elimination of Lagrangean Multipliers*, Mechanics Research Communications, vol.14, pp17-22, 1987.
39. VLASE, S, *Finite Element Analysis of the Planar Mechanisms: Numerical Aspects*. In Applied Mechanics - 4. Elsevier, 90-100, 1992.
40. VLASE, S, TEODORESCU, P.P., *Elasto-dynamics of a solid with a general "rigid" motion using FEM model. Part I.*Rom. Journ. Phys., 58(7–8), 872-881, Bucharest, 2013.
41. VLASE, S, DANASEL, C, SCUTARU, ML, MIHALCICA, M, *Finite element analysis of two-dimensional linear elastic systems with a plane "Rigid motion"*, Romanian Journal of Physics, 59(5-6), 476-487, 2014.
42. VLASE, S, TEODORESCU, PP, ITU, C, SCUTARU, M.L., *Elasto-dynamics of a solid with a general "rigid" motion using FEM model. Part II.* Analysis of a double cardan joint. Romanian Journal of Physics, 58(7-8), 882-892, 2013.
43. VLASE, S., MARIN, M., ÖCHSNER, A., SCUTARU, M.L., *Motion equation for a flexible one-dimensional element used in the dynamical analysis of a multibody system*. Continuum Mechanics and Thermodynamics, DOI: 10.1007/s00161-018-0722-y, p.1-10, 2018.
44. VLASE, S., MARIN, M., ÖCHSNER, A., *Eigenvalue and Eigenvector Problems in Applied Mechanics*. Springer, 2019
45. ZHANG, X., ERDMAN, A.G., *Dynamic responses of flexible linkage mechanisms with viscoelastic constrained layer damping treatment*. Computers & Structures, 79(13), 1265–1274, 2001.
46. ZHANG, X., LU, J, SHEN, Y., *Simultaneous optimal structure and control design of flexible linkage mechanism for noise attenuation*, Journal of Sound and Vibration, 299(4–5), 1124–1133, 2007.

Received September 24, 2018